69
9

HIGH SCHOOL MATHEMATICS

Unit 9.

ELEMENTARY FUNCTIONS:
POWERS, EXPONENTIALS, AND LOGARITHMS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

UNIVERSITY OF ILLINOIS PRESS · URBANA, 1962

HIGH SCHOOL MATHEMATICS

Unit 9.

ELEMENTARY FUNCTIONS:
POWERS, EXPONENTIALS, AND LOGARITHMS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*
HERBERT E. VAUGHAN, *Editor*

UNIVERSITY OF ILLINOIS PRESS · URBANA, 1962

PREFACE

There is a number b such that $10^b = 100$. Do you think that there is a number a such that $10^a = 75$? A number c such that $10^c = 125$? If there are such numbers, what can you say about a, b, and c?

Here is a problem whose solution depends on being able to find such exponents:

> A certain isotope of the element strontium [Sr^{89}] decays [into yttrium eighty nine] according to the formula:
>
> $$A(t) = A \cdot 10^{-2t}$$
>
> [Here, for each t, A(t) is the number of grams of the isotope which remains in a sample which, t years earlier, contained A grams.] A sample which now contains, say, 100 gm of yttrium, contains 1 gm of Sr^{89}. Supposing that all of the yttrium resulted from the decay of Sr^{89}, how many years ago did the sample contain 75 gm of the isotope?

To solve this problem, we need to find the number t such that

$$1 = 75 \cdot 10^{-2t}$$

--that is, such that $10^{2t} = 75.$

Evidently, $t = a/2$, where a is the number such that $10^a = 75$.

One of the things you will learn in this unit is how to solve such equations.

In Unit 8 you learned the meaning of expressions such as:

$$10^2, \quad 10^{-1}, \quad (a^2 + 3)^{-2}, \quad (b + 5)^{59},$$

and established some useful laws of exponents [Theorems 152a and 154-161]. In this unit you will discover how to assign meanings to expressions like:

$$10^{1.87506}, \quad 10^{-\frac{3}{4}}, \quad 5^{\sqrt{2}}, \quad \text{and:} \quad (a^{\pi} - 3^{0.8})^{\frac{2}{3}},$$

in such a way that these laws of exponents still hold. To do this, you will need to learn one more basic principle for real numbers--the least upper bound principle. [Except for definitions, this is the last basic principle you will need in order to develop all the properties of the real numbers.]

TABLE OF CONTENTS

9.01 <u>Definite description.</u> --Consider these two questions about the greatest integer function:

(1) What is <u>the real number z such that $[\![z]\!] = 1.5$</u>?

(2) What is <u>the real number z such that $[\![z]\!] = 9$</u>?

In order to answer either of these questions, you would have to name some real number.

The first question is unanswerable because 1.5 is not an integer. The second question is also unanswerable. Why?

The underlined phrases are examples of <u>definite</u> <u>descriptions</u>. [A definite description is a phrase of the form '<u>the</u> ... such that ___'; a phrase of the form '<u>a</u> ... such that ___' is an <u>in</u>definite description.] For a definite description to make sense it must be the case that

(i) there <u>is</u> <u>something</u> which satisfies the description, and

(ii) there are <u>not</u> <u>two</u> <u>things</u> which satisfy the description.

[Of course, if there are more than two, there are two.]

PRINCIPAL SQUARE ROOTS

Consider these questions about the squaring function:

(3) What is the real number z such that $z^2 = 4$?

(4) What is the real number z such that $z^2 = -1$?

(5) What is the real number z such that $z^2 = \pi$?

To answer any of these questions you must name some real number.

Question (3) is unanswerable because there are <u>two</u> real numbers [2 and -2] whose squares are 4. [If <u>each</u> of these numbers were <u>the</u> number whose square is 4, then they would not be two numbers! In other words, it would be the case that $2 = -2$.]

Question (4) is unanswerable because squares of real numbers are nonnegative. [Theorem 97a and the pm0.]

Question (5) is also unanswerable. In the first place, as far as anything we have proved is concerned, there may not be a real number whose square is π. But, if there is, there are two [Explain.].

Let's replace question (3) by a related question which is answerable:

(3') What is the real number z such that ($z \geq 0$ and $z^2 = 4$)?

What is the answer?

If we make a similar change in question (4), we still have an unanswerable question. Why?

If we make a similar change in question (5), the resulting question is answerable if and only if there is a real number whose square is π. Explain.

Let's go back to question (3′) and see in more detail why it is answerable. In order for the definite description:

the real number z such that ($z \geq 0$ and $z^2 = 4$)

to make sense, the description must refer to exactly one real number. In other words, it must be the case that two conditions are satisfied--

<u>an existence condition</u>:

There is a number which is nonnegative and whose square is 4.

and

<u>a uniqueness condition</u>:

There are not two numbers which are nonnegative and whose squares are 4.

Since $2 \geq 0$ and $2^2 = 4$, the existence condition is satisfied. Let's consider the uniqueness condition. Suppose that ($b \geq 0$ and $b^2 = 4$) and ($c \geq 0$ and $c^2 = 4$), and suppose that $b \neq c$. From the last assumption it follows that

$$c > b \quad \text{or} \quad b > c. \qquad \text{[Theorem 86a]}$$

Now, since $b \geq 0$,

$$\text{if } c > b \text{ then } c^2 > b^2$$

and, since $c \geq 0$, [Theorem 98c]

$$\text{if } b > c \text{ then } b^2 > c^2.$$

Hence,

$$c^2 > b^2 \quad \text{or} \quad b^2 > c^2.$$

In either case [by Theorem 87], $b^2 \neq c^2$. Consequently,

$$\text{if } b \neq c \text{ then } b^2 \neq c^2.$$

But, by hypothesis, $b^2 = 4 = c^2$. Hence, $b = c$. There are not <u>two</u> numbers which are nonnegative and whose squares are 4.

As you may recall from earlier units, we called the nonnegative number whose square is 4

<u>the principal square root of 4</u>

and abbreviated this to:

$$\sqrt{4}$$

[You may also read this as 'radical 4'.]

In general,

$$(*) \quad \forall_{x \geq 0} \sqrt{x} = \text{the real number } z \text{ such that } (z \geq 0 \text{ and } z^2 = x).$$

[Why '$x \geq 0$' in the quantifier?]

As in the case $x = 4$, in order for the descriptive phrase in (*) to make sense, two conditions must be satisfied--an existence condition:

$$(\dagger_1) \quad \forall_{x \geq 0} \exists_z (z \geq 0 \text{ and } z^2 = x)$$

and a uniqueness condition:

$$(\dagger_2) \quad \forall_{x \geq 0} \forall_y \forall_z [((y \geq 0 \text{ and } y^2 = x) \text{ and } (z \geq 0 \text{ and } z^2 = x)) \Rightarrow y = z]$$

[Compare the instances of $(\dagger_1)$ and $(\dagger_2)$ for $x = 4$ with the existence condition and the uniqueness condition on page 9-2.] Again, as in the case $x = 4$, we need to ask whether $(\dagger_1)$ and $(\dagger_2)$ are satisfied. Let's begin with $(\dagger_1)$.

We know that the instance of $(\dagger_1)$ for $x = 4$:

$$\exists_z (z \geq 0 \text{ and } z^2 = 4)$$

is satisfied because we can <u>prove</u> that $2 \geq 0$ and $2^2 = 4$. And, in a similar way, we can establish many other instances of $(\dagger_2)$. But, consider the instance:

$$(1) \qquad \exists_z (z \geq 0 \text{ and } z^2 = 8)$$

A few attempts will show you that this is not so easy to prove. [Note that it is not fair, at this point, to answer that, by definition, $\sqrt{8}$ is a nonnegative number whose square is 8. It is precisely in order to help show that '$\sqrt{8}$' makes sense that we need to prove the statement (1).]

As a matter of fact, it is not hard to show that the instance (1) of $(\dagger_1)$ is <u>not</u> a consequence of the basic principles which we have adopted so far. [This is done on page 9-23.] Consequently, the same holds of

the existence condition $(\dagger_1)$. So, if we are to be able to prove theorems about principal square roots of all nonnegative numbers, we shall need a new basic principle. We might take $(\dagger_1)$, itself:

$$(\dagger_1)\quad \forall_{x \geq 0} \exists_z\, (z \geq 0 \text{ and } z^2 = x)$$

as such a basic principle and, in fact, this is what we did, implicitly, when we dealt informally with square roots in earlier units. However, this is a shortsighted policy. As we shall see, we shall want many theorems like $(\dagger_1)$--for example:

$$\forall_x \exists_y\, y^3 = x, \qquad \text{and:} \qquad \forall_{x>0} \exists_y\, 2^y = x$$

--and it is more economical to find a single basic principle which will allow us to prove all the theorems of this kind which we want. You will learn one such principle in section 9. 03.

Now, let's consider the uniqueness condition:

$$(\dagger_2)\quad \forall_{x \geq 0} \forall_y \forall_z [((y \geq 0 \text{ and } y^2 = x) \text{ and } (z \geq 0 \text{ and } z^2 = x)) \Rightarrow y = z]$$

Can we prove this now, or do we need still another basic principle? Fortunately, $(\dagger_2)$ is already a theorem. This should be clear from the proof, on page 9-2, of its instance for x = 4. To obtain a test-pattern for $(\dagger_2)$, all we need do is replace the '4's in this proof by, say, 'a's. [Notice that the proof of $(\dagger_2)$ used only three theorems--two general theorems about order and one theorem [Theorem 98c] relating to squaring. The other uniqueness conditions which we shall need we shall be able to prove in just the same way, using the same two theorems about order and, in each case, one theorem [like Theorem 98c] relating to an appropriate function--for example, the cubing function and the exponential function with base 2.]

As remarked above, the existence condition $(\dagger_1)$ can be proved with the help of one new basic principle, and the uniqueness condition $(\dagger_2)$ is already a theorem. Consequently [assuming that we have already adopted this new basic principle], we are entitled to speak, for any $a \geq 0$, of <u>the</u> real number z which is nonnegative and whose square is the number a. Rather than use such a lengthy description, we should prefer to abbreviate it, as suggested in (*), by means of the operator '$\sqrt{}$'. However, if we are to prove theorems which contain this new operator, we need

to have at least one basic principle in which the operator occurs. For this purpose, we shall adopt a defining principle for '$\sqrt{\ }$':

$$(*_1) \qquad \forall_{x \geq 0} (\sqrt{x} \geq 0 \text{ and } (\sqrt{x})^2 = x)$$

[Compare $(*_1)$ with $(\dagger_1)$.] The theorems $(\dagger_1)$ and $(\dagger_2)$ tell us that, for any $a \geq 0$, there is one and only one real number z such that ($z \geq 0$ and $z^2 = a$). The defining principle $(*_1)$, then, gives notice that this real number z is $\sqrt{a}$.

Once we have adopted the defining principle $(*_1)$, we can use it and $(\dagger_2)$ to obtain a useful complement to $(*_1)$. This is the uniqueness theorem for '$\sqrt{\ }$':

$$(*_2) \qquad \forall_{x \geq 0} \forall_y [(y \geq 0 \text{ and } y^2 = x) \Rightarrow y = \sqrt{x}]$$

To prove $(*_2)$, we proceed as follows: Suppose that $b \geq 0$ and $b^2 = a$. By $(*_1)$, for $a \geq 0$, ($\sqrt{a} \geq 0$ and $(\sqrt{a})^2 = a$). By $(\dagger_2)$, for $a \geq 0$,

$$\text{if } (b \geq 0 \text{ and } b^2 = a) \text{ and } (\sqrt{a} \geq 0 \text{ and } (\sqrt{a})^2 = a) \text{ then } b = \sqrt{a}.$$

So, by the assumption and $(*_1)$, it follows that $b = \sqrt{a}$. Hence, for $a \geq 0$,

$$\text{if } b \geq 0 \text{ and } b^2 = a \text{ then } b = \sqrt{a}.$$

Consequently, $\forall_{x \geq 0} \forall_y [(y \geq 0 \text{ and } y^2 = x) \Rightarrow y = \sqrt{x}]$.

The generalizations $(*_1)$ and $(*_2)$ tell us all we need to know about the operator '$\sqrt{\ }$'. To illustrate this, let's recall a typical proof about principal square roots--the proof of:

$$\forall_{x \geq 0} \forall_{y \geq 0} \sqrt{x}\sqrt{y} = \sqrt{xy}$$

For $a \geq 0$ and $b \geq 0$, it follows from $(*_1)$ that $\sqrt{a} \geq 0$, $(\sqrt{a})^2 = a$, $\sqrt{b} \geq 0$, and $(\sqrt{b})^2 = b$. Since $\sqrt{a} \geq 0$ and $\sqrt{b} \geq 0$, $\sqrt{a}\sqrt{b} \geq 0$. Moreover, $(\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2$ and, so, since $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$, $(\sqrt{a}\sqrt{b})^2 = ab$.

Now, since, for $a \geq 0$ and $b \geq 0$, $ab \geq 0$ and since, for $a \geq 0$ and $b \geq 0$, $\sqrt{a}\sqrt{b} \geq 0$ and $(\sqrt{a}\sqrt{b})^2 = ab$, it follows from $(*_2)$ that $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

Consequently, $\forall_{x \geq 0} \forall_{y \geq 0} \sqrt{x}\sqrt{y} = \sqrt{xy}$.

In the following exercises, assume that $(*_1)$ is either a theorem or [temporarily] a basic principle, and that, consequently, $(*_2)$ is a theorem.

EXERCISES

A. Prove.

1. $\sqrt{1} = 1$ 2. $\sqrt{0} = 0$ 3. $(\sqrt{\pi})^2 = \pi$

B. Prove, or give a counter-example.

1. $\forall_x \sqrt{x^2} = x$ 2. $\forall_{x \geq 0} \sqrt{x^2} = x$ 3. $\forall_{x \leq 0} \sqrt{x^2} = -x$

4. $\forall_x \sqrt{x^2} = |x|$ [Hint. $\forall_{x \geq 0}\ |x| = x$ and $\forall_{x \leq 0}\ |x| = -x$]

C. True or false?

1. $\forall_x \sqrt{x^4} = x^2$ 2. $\forall_x \sqrt{x^6} = x^3$ 3. $\forall_x \sqrt{x^6} = -x^3$

4. $\forall_n \forall_x \sqrt{x^{2n}} = x^n$ 5. $\forall_n \forall_x \sqrt{x^{2n}} = |x|^n$

6. $\forall_n \forall_x \sqrt{x^{4n}} = x^{2n}$ 7. $\forall_x \forall_y \sqrt{x^4 y^8} = x^2 y^4$

8. $\forall_x \forall_y \sqrt{x^6 y^{10}} = x^3 y^5$ 9. $\sqrt{(2-1)^2} = 2 - 1$

10. $\sqrt{(1-2)^2} = 1 - 2$ 11. $\sqrt{(5-3)^2} = |5 - 3|$

12. $\sqrt{(3-5)^2} = |3 - 5|$ 13. $\sqrt{-9 \times -4} = -3 \times -2$

14. $\sqrt{9 \times 4} = -3 \times -2$ 15. $\forall_x \sqrt{(x-3)^2} = x - 3$

16. $\sqrt{(2-3)^2} = 2 - 3$ 17. $\forall_{x \geq 3} \sqrt{(x-3)^2} = x - 3$

18. $\forall_{x < 3} \sqrt{(x-3)^2} = 3 - x$ 19. $\forall_x \sqrt{(x-3)^2} = |x - 3|$

20. $\forall_x \sqrt{x^2 + 6x + 9} = x + 3$ 21. $\sqrt{9 + 16} = 3 + 4$

D. Practice manipulating square root expressions.

Sample 1. Simplify: $\sqrt{50} + \sqrt{18}$

Solution. $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$,

$\sqrt{18} = \sqrt{? \cdot 2} = \sqrt{?} \cdot \sqrt{2} = 3\sqrt{2}$,

$\sqrt{50} = \sqrt{18} = 5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$

Sample 2. Simplify: $3\sqrt{50} + 4\sqrt{12} - \sqrt{8} + \sqrt{48}$

Solution. $3\sqrt{50} + 4\sqrt{12} - \sqrt{8} + \sqrt{48}$

$= 15\sqrt{2} + 8\sqrt{3} - 2\sqrt{2} + 4\sqrt{3}$

$= 13\sqrt{2} + 12\sqrt{3}$

Simplify.

1. $\sqrt{72} + \sqrt{8}$
2. $\sqrt{12} + \sqrt{75}$
3. $\sqrt{125} - \sqrt{20}$
4. $5\sqrt{108} + 2\sqrt{243} - \sqrt{27} + 2\sqrt{12}$
5. $\sqrt{20} - \sqrt{24} + \sqrt{125} - \sqrt{54}$
6. $\frac{1}{2}\sqrt{24} - \frac{1}{3}\sqrt{54} + \frac{1}{4}\sqrt{6} - \frac{1}{2}\sqrt{150}$
7. $\sqrt{0.1} + 3\sqrt{0.01} + 10\sqrt{0.001}$
8. $\sqrt{144} + \sqrt{1.44} + \sqrt{0.0144}$
9. $\sqrt{14400} + \sqrt{144} + \sqrt{0.000144}$
10. $\sqrt{200} + \sqrt{2} + \sqrt{0.02}$
11. $\sqrt{2000} + \sqrt{200} + \sqrt{20} + \sqrt{2}$

E. True or false?

1. $2 > \sqrt{2}$
2. $0.01 > \sqrt{0.01}$
3. $\pi > \sqrt{\pi}$
4. $0.1 > \sqrt{0.1}$
5. $1 > \sqrt{1}$
6. $0.999 > \sqrt{0.999}$

F. More practice manipulating square root expressions.

Sample 1. Simplify: $\dfrac{\sqrt{18}}{\sqrt{2}}$

Solution. $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$

Sample 2. Simplify: $\sqrt{3}\sqrt{7}$

Solution. $\sqrt{3}\sqrt{7} = \sqrt{3 \cdot 7} = \sqrt{21}$

Sample 3. Simplify: $(\sqrt{2})^3$

Solution. $(\sqrt{2})^3 = (\sqrt{2})^2\sqrt{2} = 2\sqrt{2}$

Simplify.

1. $\dfrac{\sqrt{21}}{\sqrt{7}}$
2. $\dfrac{5\sqrt{6}}{\sqrt{2}}$
3. $\dfrac{8\sqrt{10}}{\sqrt{8}}$
4. $\sqrt{\frac{1}{4}}\sqrt{12}$

5. $\sqrt{2}\sqrt{5}$ 6. $\sqrt{\frac{2}{3}}\sqrt{\frac{15}{2}}$ 7. $\sqrt{8}\sqrt{0.125}$ 8. $\frac{8\sqrt{5}}{\sqrt{20}}$

9. $(\sqrt{3})^3$ 10. $(\sqrt{0.01})^4$ 11. $(\sqrt{0.1})^4$ 12. $(-\sqrt{\pi})^3$

13. $\left(\frac{\sqrt{2}}{2}\right)^3$ 14. $\left(\frac{\sqrt{2}}{2}\right)^{-2}$ 15. $\left(\frac{\sqrt{3}}{2}\right)^{-4}$ 16. $(0.01\sqrt{5})^2$

17. $\frac{2}{\sqrt{2}}$ 18. $\left(\frac{3}{\sqrt{3}}\right)^2$ 19. $\left(\frac{-5}{\sqrt{5}}\right)^3$ 20. $\left(\frac{4}{\sqrt{4}}\right)^{-3}$

G. Expand.

Sample. $\left(\frac{4+\sqrt{2}}{2}\right)^2$

Solution. $\left(\frac{4+\sqrt{2}}{2}\right)^2 = \frac{16+8\sqrt{2}+2}{4} = \frac{9+4\sqrt{2}}{2}$

1. $(3+\sqrt{2})^2$ 2. $(4-\sqrt{5})^2$ 3. $(2+\sqrt{3})^3$

4. $(\sqrt{2}-1)^4$ 5. $(\sqrt{2}+\sqrt{6})^2$ 6. $(\sqrt{3}-\sqrt{6})^3$

7. $(\sqrt{8}-1)(\sqrt{8}+1)$ 8. $(1+\sqrt{5})(1-\sqrt{5})$

9. $(\sqrt{2}+3)(\sqrt{2}-4)$ 10. $(3-\sqrt{7})(4+\sqrt{7})$

11. $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$ 12. $(2\sqrt{3}+3\sqrt{2})(5\sqrt{3}-2\sqrt{2})$

13. $(2\sqrt{5}-4\sqrt{7})(\sqrt{5}+2\sqrt{7})$ 14. $(\sqrt{12}+\sqrt{18})(\sqrt{75}-\sqrt{8})$

15. $(3\sqrt{7}+2)^2$ 16. $(5\sqrt{3}-2)^2$ 17. $(\sqrt{3}-\sqrt{2})^2$

18. $(x-\sqrt{y})^2$ 19. $(2a+\sqrt{b})^2$ 20. $(3\sqrt{t}-\sqrt{-s})^2$

H. Prove the following theorems. [Use $(*_1)$, $(*_2)$, and earlier theorems.]

1. $\forall_{x\geq 0}\forall_{y>0}\ \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ 2. $\forall_{x\geq 0}\forall_{y>0}\ \sqrt{\frac{x}{y}} = \frac{\sqrt{xy}}{y}$

*

Complete.

3. $\forall_{x\leq 0}\forall_{y\leq 0}\ \sqrt{xy} = \sqrt{\quad}\cdot\sqrt{\quad}$ 4. $\forall_{x\leq 0}\forall_{y<0}\ \sqrt{\frac{x}{y}} = \frac{\sqrt{\quad}}{\sqrt{\quad}}$

5. $\forall_{x\leq 0}\ \forall_{y<0}\ \sqrt{\frac{x}{y}} = \frac{\sqrt{xy}}{\quad}$

I. The samples below suggest various techniques for transforming expressions containing square root signs. Practice these techniques.

Sample 1. $\sqrt{50x^3}$

Solution. $\sqrt{50x^3} = \sqrt{(25x^2)x}$
$= \sqrt{(5x)^2}\sqrt{x}$
$= |5x|\sqrt{x}$
$= 5|x|\sqrt{x}$

For the expression '$\sqrt{50x^3}$' to make sense, the values of 'x' must be restricted to be nonnegative numbers. [Why?] So, the "final answer" is: $5x\sqrt{x}$ $[x \geq 0]$

Sample 2. $\sqrt{162a^7b^{-6}}$

Solution. $\sqrt{162a^7b^{-6}} = \sqrt{(9a^3b^{-3})^2(2a)}$
$= 9|a^3b^{-3}|\sqrt{2a}$

For the expression '$\sqrt{162a^7b^{-6}}$' to make sense, the values of 'a' must be restricted to be nonnegative numbers, and those of 'b' must be restricted to be nonzero numbers. So, the "final answer" is:

$$9a^3|b|^{-3}\sqrt{2a} \qquad [a \geq 0,\ b \neq 0]$$

1. $\sqrt{36y}$
2. $\sqrt{81t^3}$
3. $\sqrt{-81t^3}$
4. $\sqrt{45x^3y^2}$
5. $\sqrt{6a^4b^5}$
6. $\sqrt{0.25s^4t^8}$
7. $\sqrt{16x^{-6}}$
8. $\sqrt{a^6b^{-4}}$
9. $\sqrt{\dfrac{3ab^2c^3}{4d^2e^4}}$
10. $\sqrt{(x-y)^3}$
11. $\sqrt{a^3 + 3a^2b + 3ab^2 + b^3}$
12. $\sqrt{81x \times 64y \times 324x^2y^3}$
13. $\sqrt{15ab \times 33bc \times 55ca}$
14. $\sqrt{(x^2 + x - 6)(x^2 - 7x + 10)(x^2 - 2x - 15)}$

<u>Sample 3.</u> Transform to an expression which is not a fraction with a square root sign in the denominator--that is, <u>rationalize the denominator</u>:

$$\frac{3}{\sqrt{2}}$$

<u>Solution.</u> $\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

<u>Sample 4.</u> $\frac{2 + \sqrt{2}}{3 - \sqrt{2}}$

<u>Solution.</u>
$$\frac{2 + \sqrt{2}}{3 - \sqrt{2}} = \frac{2 + \sqrt{2}}{3 - \sqrt{2}} \cdot \frac{3 + \sqrt{2}}{3 + \sqrt{2}}$$
$$= \frac{6 + 5\sqrt{2} + 2}{9 - 2}$$
$$= \frac{8 + 5\sqrt{2}}{7}$$

<u>Sample 5.</u> $\sqrt{\frac{7}{18}}$

<u>Solution.</u> $\sqrt{\frac{7}{18}} = \sqrt{\frac{7}{18} \cdot \frac{2}{2}} = \sqrt{\frac{14}{36}} = \frac{\sqrt{14}}{6}$

15. $\frac{2}{\sqrt{3}}$ 16. $\sqrt{\frac{2}{3}}$ 17. $\frac{a}{\sqrt{b}}$ 18. $\sqrt{\frac{a}{b}}$

19. $\frac{a}{\sqrt{a}}$ 20. $\frac{5 - \sqrt{3}}{4 + \sqrt{3}}$ 21. $\frac{7 + \sqrt{3}}{2 + \sqrt{3}}$ 22. $\frac{1}{\sqrt{x} - \sqrt{y}}$

23. $\frac{\sqrt{3} + \sqrt{5}}{\sqrt{7} - \sqrt{2}}$ 24. $\frac{a + \sqrt{b}}{b + \sqrt{a}}$ 25. $\frac{3}{1 + \sqrt{2} + \sqrt{3}}$

<u>Sample 6.</u> Find the rational approximation to $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ correct to the nearest tenth.

<u>Solution.</u> $\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \doteq \frac{1.732 + 1}{1.732 - 1} = \frac{2.732}{0.732} = \frac{2732}{732} = ?$

If you have a desk calculator available, the problem is easy to finish. But, if you wish to avoid doing "long division", it would be easier to transform the given expression as in the previous exercises:

$$\begin{aligned}\frac{\sqrt{3}+1}{\sqrt{3}-1} &= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}\\ &= \frac{4+2\sqrt{3}}{2}\\ &= 2+\sqrt{3}\\ &\doteq 2+1.732\\ &\doteq 3.7\end{aligned}$$

Find rational approximations correct to the nearest tenth. [Use the table of square roots on page 9-365.]

26. $\sqrt{\frac{8}{7}}$

27. $\frac{\sqrt{5}+2}{\sqrt{5}-2}$

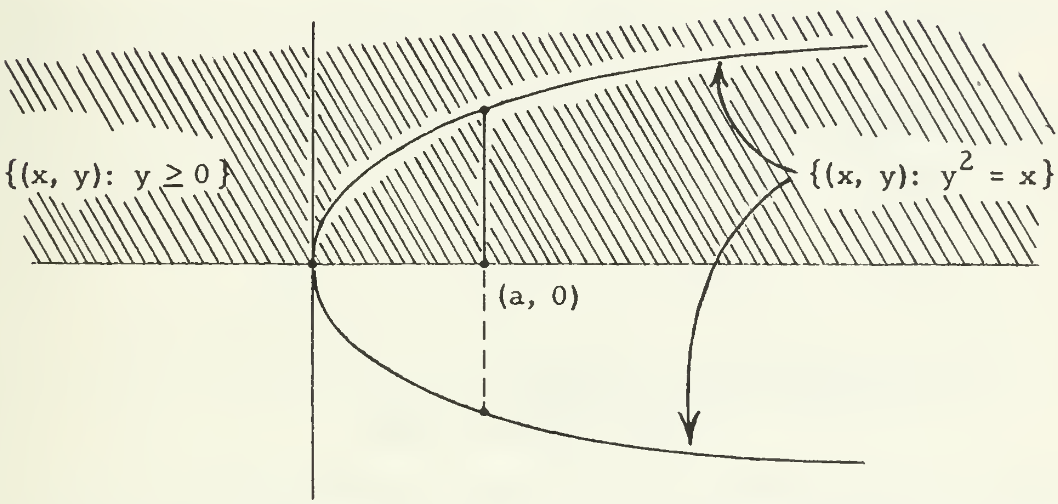
28. $\frac{5}{\sqrt{6}-\sqrt{3}}$

29. $\frac{\sqrt{15}-\sqrt{2}}{\sqrt{15}-\sqrt{10}}$

30. $\frac{1+\sqrt{2}}{\sqrt{7}-\sqrt{5}}$

31. $\frac{\sqrt{\frac{1}{2}+\frac{1}{3}}}{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}$

J. Here are graphs of '$y^2 = x$' and '$y \geq 0$'.



The existence and uniqueness conditions which justify our adopting the defining principle for principal square roots are:

($\dagger_1$) $\forall_{x \geq 0} \exists_z (z \geq 0 \text{ and } z^2 = x)$

($\dagger_2$) $\forall_{x \geq 0} \forall_y \forall_z [(y \geq 0 \text{ and } y^2 = x) \text{ and } (z \geq 0 \text{ and } z^2 = x)) \Rightarrow y = z]$

1. How does the figure suggest what ($\dagger_1$) says?

2. How does the figure suggest what ($\dagger_2$) says?

K. The cube root operator '$\sqrt[3]{\ }$' is defined by:

$$(*') \qquad \forall_x \sqrt[3]{x} = \text{the } z \text{ such that } z^3 = x$$

1. State the existence and uniqueness conditions which justify the use of the definite description occurring in (*'):

$(t_1') \qquad \forall_x \exists_z \ldots$

$(t_2') \qquad \forall_x \forall_y \forall_z \ldots$

2. Draw a graph of '$y^3 = x$' and point out the properties of the graph that suggest what (t_1') says and what (t_2') says.

3. Complete.

(a) $\sqrt[3]{1} =$ (b) $\sqrt[3]{8} =$ (c) $\sqrt[3]{-8} =$ (d) $\sqrt[3]{8/27} =$

(e) $\sqrt[3]{2}\,\sqrt[3]{4} =$ (f) $\sqrt[3]{4}\,\sqrt[3]{-16} =$ (g) $\sqrt[3]{40} = 2\sqrt[3]{\ }$

4. State the defining principle and the uniqueness theorem for the cube root operator which bear the same relation to (*') as $(*_1)$ and $(*_2)$ on page 9-5 bear to (*) on page 9-3:

$(*_1') \qquad \forall_x (\quad)^3 =$

$(*_2') \qquad \forall_x \forall_y \ldots$

5. Use $(*_1')$ and $(*_2')$ in proving:

$$\forall_x \forall_y \sqrt[3]{x}\,\sqrt[3]{y} = \sqrt[3]{xy}$$

L. 1. The adoption of a descriptive definition like (*) commits one to two statements like $(*_1)$ and $(*_2)$. For example, if one were to adopt:

$$\forall_x \sqrt{x} = \text{the real number } z \text{ such that } (z \geq 0 \text{ and } z^2 = x)$$

he would be committed to:

$$\forall_x (\sqrt{x} \geq 0 \text{ and } (\sqrt{x})^2 = x)$$

So, in particular, he would be forced to accept a real number

$\sqrt{-1}$ such that $\sqrt{-1} \geq 0$ and $(\sqrt{-1})^2 = -1$.

Show that, because of a previous theorem, this would be an unfortunate position to be in.

2. Adoption, as a definition, of:

$$\forall_{x \geq 0} \sqrt{x} = \text{the real number } z \text{ such that } z^2 = x$$

would commit one to the statement:

$$\forall_{x \geq 0} \forall_y [y^2 = x \Rightarrow y = \sqrt{x}]$$

Use the fact that $4 \geq 0$, $2^2 = 4$, and $(-2)^2 = 4$ in showing that this commitment would be an unfortunate one.

*

The two exercises of Part L point up the fact that it is desirable to be aware of one's commitments--in particular, to be aware of what basic principles he reasons from.

*

The rest of the exercises--Parts ☆M, ☆N, and ☆O--are optional.

☆M. Here are an existence condition and a uniqueness condition which, since they are theorems, justify the introduction of an operator [you studied this operator in Unit 7]:

$$\forall_x \exists_z \ (z \in I \text{ and } z \leq x < z + 1)$$

$$\forall_x \forall_y \forall_z [((y \in I \text{ and } y \leq x < y + 1) \text{ and } (z \in I \text{ and } z \leq x < z + 1)) \Rightarrow y = z]$$

1. Let's consider how you might think of a proof for the existence condition. Given a number a, you are interested in the set of those integers which are less than or equal to a. The existence condition refers to a greatest such integer. Your problem is to prove that there is a greatest integer z such that $z \leq a$. What existence theorem about integers might be helpful in proving this?

2. The uniqueness condition can be proved by deriving it from three theorems--one like Theorem 92, and Theorems 112 and 93. Do so.

3. What is the operator?

4. State the defining principle and the uniqueness theorem for this operator:

$$\forall_x \, ($$

$$\forall_x \forall_y \, [\qquad\qquad \Rightarrow \qquad\qquad]$$

5. The uniqueness theorem can be derived from the defining principle by slightly modifying the proof you gave in answer to Exercise 2. Make these modifications.

☆6. (a) It is a trivial task to show that the existence condition is a consequence of the defining principle. Do this.

(b) It is almost as trivial a task to show that the uniqueness condition is a consequence of the uniqueness theorem. Do this.

*

A defining principle for the absolute value operator is:

$$(*_1'') \qquad \forall_x \, (|x| \geq 0 \text{ and } (|x| = x \text{ or } |x| = -x))$$

*

☆M. [continued]

7. State the existence and uniqueness conditions which would justify adopting the defining principle given above.

8. Using the principle for multiplying by 0 and the 0-product theorem it is easy to prove:

$$\forall_u \forall_v \, [(u = 0 \text{ or } v = 0) \Longleftrightarrow uv = 0]$$

This gives one a way of replacing alternation sentences of a certain kind by sentences which do not contain the word 'or'. Use this method to restate, without using 'or', the defining principle given just before Exercise 7.

9. Use the restatement you gave in answer to Exercise 8 together with the uniqueness theorem ($*_2$) for square roots [page 9-5] to give a quick proof of:

$$\forall_x \ |x| = \sqrt{x^2}$$

[What other theorem did you need to prepare the way for using ($*_2$)?]

☆N. In Part ☆M, above, we gave a defining principle for the operator '| |':

$$(*_1'') \quad \forall_x \ (|x| \geq 0 \text{ and } (|x| = x \text{ or } |x| = -x))$$

In the following exercises and those of Part ☆O you will see how this principle can be used to prove theorems about absolute values. [Most of these theorems you may already have proved, in another way, in Unit 8. Some of them are collected in Theorem 169.] As an example of how this defining principle is used, let's prove:

Theorem A.
$$\forall_x \forall_y \ [-y \leq x \leq y \Rightarrow |x| \leq y]$$

Suppose that $-b \leq a \leq b$. By the second part of the defining principle, $|a| = a$ or $|a| = -a$. Suppose that $|a| = a$. In this case, it follows from our initial assumption that $|a| \leq b$. Suppose that $|a| = -a$. Since, from our initial assumption, $-a \leq b$ [Why?], it follows, in this case also, that $|a| \leq b$. Hence, in any case, if $-b \leq a \leq b$ then $|a| \leq b$. Consequently, $\forall_x \forall_y \ [-y \leq x \leq y \Rightarrow |x| \leq y]$.

The same technique can be used to prove:

Theorem B. $\forall_x \forall_y \ [-y < x < y \Rightarrow |x| < y]$

1. Complete.

Sample 1. $-5 < a < 5 \Rightarrow$

Solution. $-5 < a < 5 \Rightarrow |a| < 5$ [Theorem B]

Sample 2. $3 \le a \le 5 \Rightarrow$

Solution. $3 \le a \le 5 \Leftrightarrow 3 - 4 \le a - 4 \le 5 - 4$

$\Leftrightarrow -1 \le a - 4 \le 1$

$\Rightarrow |a-4| \le 1$ [Theorem A]

[Note that 4 is the midpoint of the segment $\overleftrightarrow{3, 5}$.]

(a) $-8 < a < 8 \Rightarrow$ (b) $-2 \le a \le 2 \Rightarrow$

(c) $9 < a < 13 \Rightarrow$ (d) $-6 \le a \le 0 \Rightarrow$

(e) $5 < a < 6 \Rightarrow$ (f) $2 \le a + 5 \le 9 \Rightarrow$

*

The converses of Theorems A and B are also theorems. To prove them it is useful to have a lemma:

$$\forall_x \; -|x| \le x \le |x|$$

Let's prove this.

By the first part of $(*_1'')$, $|a| \ge 0$. Hence, $0 \le |a| + |a|$. Consequently, $-|a| \le |a|$. Now, by the second part of $(*_1'')$, $a = |a|$ or $a = -|a|$. In the first case,

$$-|a| \le a = |a|.$$

In the second case,

$$-|a| = a \le |a|.$$

So, in either case, $-|a| \le a \le |a|$. [Explain.]

Now let's prove the converse of Theorem B:

Theorem C.

$$\forall_x \forall_y \, [|x| < y \Rightarrow -y < x < y]$$

Suppose that $|a| < b$. Since, by the lemma, $a \le |a|$, it follows that $a < b$. On the other hand, since $|a| < b$, it follows that $-b < -|a|$, and since, from the lemma, $-|a| \le a$, that $-b < a$. Hence, if $|a| < b$ then $-b < a < b$.

[The converse of Theorem A can be proved by the same technique.]

Theorems B and C give us the following biconditional:

> <u>Theorem D.</u>
>
> $\forall_x \forall_y\, [\,|x| < y \Longleftrightarrow -y < x < y]$

*

2. Complete.

<u>Sample</u>. $|a + 3| < 0.5 \Longleftrightarrow$

<u>Solution</u>. $|a + 3| < 0.5 \Longleftrightarrow -0.5 < a + 3 < 0.5$ [Thm. D.]

$\Longleftrightarrow -3.5 < a < -2.5$

(a) $|a| < 0.1 \Longleftrightarrow$

(b) $|a + 2| < 0.05 \Longleftrightarrow$

(c) $|a - 3| < 0.01 \Longleftrightarrow$

(d) $|a - 3| < d \Longleftrightarrow$

(e) $|a + 2| < d \Longleftrightarrow$

(f) $|a - a_o| < d \Longleftrightarrow$

(g) $|3a - 5| < 1 \Longleftrightarrow$

(h) $|3a - 5| < c \Longleftrightarrow$

*

As a consequence of the lemma for Theorem C,

$$-(|a| + |b|) \le a + b \le |a| + |b|, \qquad \text{[Explain.]}$$

and, from this, by Theorem A,

$$|a + b| \le |a| + |b|.$$

So, we have proved:

> <u>Theorem E.</u> $\forall_x \forall_y\ |x + y| \le |x| + |y|$

*

3. Use Theorem E to prove:

> <u>Theorem F.</u> $\forall_x \forall_y\ |x| - |y| \le |x - y|$

[<u>Hint</u>. 'x - y' for 'x' in Theorem E.]

☆O. The defining principle of Part ☆N is supplemented by a uniqueness theorem:

$$(*_2'') \quad \forall_x \forall_y\ [(y \geq 0 \text{ and } (y = x \text{ or } y = -x)) \Rightarrow y = |x|]$$

Here is a proof of this uniqueness theorem.

> Suppose that $b \geq 0$ and $(b = a$ or $b = -a)$. By the defining principle $(*_1'')$ of Part N, either $|a| = a$ or $|a| = -a$. In either case, since $b = a$ or $b = -a$, it follows that $(b = |a|$ or $b = -|a|)$. Suppose that $b = -|a|$. Since $b \geq 0$, it follows that $-|a| \geq 0$-- that is, that $|a| \leq 0$. Since, by $(*_1'')$, $|a| \geq 0$, it follows that $|a| = 0$. Since $b = -|a|$, it follows that $b = 0$, also, and, so, that $b = |a|$. So, in either case, $b = |a|$. Hence, if $b \geq 0$ and $(b = a$ or $b = -a)$ then $b = |a|$.
> Consequently, $\forall_x \forall_y\ [(y \geq 0 \text{ and } (y = x \text{ or } y = -x)) \Rightarrow y = |x|]$.

As an example of how $(*_2'')$ is used, let's prove:

$$\boxed{\underline{\text{Theorem G.}}\quad \forall_x\ |-x| = |x|}$$

According to $(*_2'')$, what we need to prove is that, for any a, $(|-a| \geq 0$ and $(|-a| = a$ or $|-a| = -a))$. That this is the case follows from $(*_1'')$ and Theorem 17.

1. Use Theorems F and G to prove:

$$\boxed{\forall_x \forall_y\ |x| - |y| \leq |x + y|}$$

2. Use Theorem A and the theorem proved in Exercise 1 to prove:

$$\boxed{\underline{\text{Theorem H.}}\quad \forall_x \forall_y\ \big||x| - |y|\big| \leq |x + y|}$$

3. Use $(*_1'')$ and $(*_2'')$ to prove:

$$\boxed{\underline{\text{Theorem I.}}\quad \forall_x \forall_y\ |x| \cdot |y| = |xy|}$$

MISCELLANEOUS EXERCISES

1. The sum of the first five terms of an arithmetic progression is one fourth the sum of the next five. What are the first two terms?

2. Solve these equations.

   (a) $84 + (a + 4)(a - 3)(a + 5) = (a + 1)(a + 2)(a + 3)$

   (b) $y - \left(3y - \frac{2y - 5}{10}\right) = \frac{1}{6}(2y - 57) - \frac{5}{3}$

3. If the sum of the first four terms of a geometric progression is 40 and the sum of the next four terms is 3240, find the first and second terms.

4. Factor.

   (a) $acx^2 - bcx + adx - bd$ (b) $(27x + y)^2 - (26x - y)^2$

   (c) $(x + y)^4 - 1$ (d) $(x + y)^3 + (x - y)^3$ (e) $a^3 + b^3 + a + b$

   (f) $x^2 - y^2 + x - y$ (g) $k^2 - k - 2$ (h) $t^2 - t - 6$

   (i) $a^3 + 27b^3$ (j) $x^3 - 64y^3$

5. The measures of the sides of a triangle are 13, 14, and 15. Use the formula:

$$K = \sqrt{s(s - a)(s - b)(s - c)}$$

   to find the area-measure of the triangle, and then find the sine ratios for the three angles.

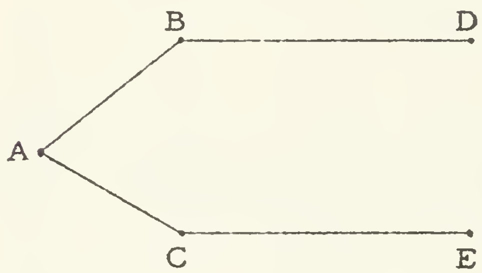
6. Suppose that A, B, and C are three points on a circle of diameter d such that $\angle BAC$ is acute. Show that the sine ratio for $\angle BAC$ is BC/d.

7. Find a root of the equation: $\sqrt[3]{x} = 10\sqrt[4]{x}$

8. Suppose that ℓ is a line and P is a point not on ℓ. If A_1, A_2, A_3, ..., A_n are n points on ℓ, what is the total number of triangles with P as one vertex and two of the points A_1, A_2, A_3, ..., A_n as the other vertices?

9. Derive a formula for the area-measure A of a regular hexagon in terms of its perimeter P.

10. Five boys hire a ping-pong table for 1 hour. Only 2 boys play at a time. If each boy is to play the same amount of time, how many minutes can each play?

11. 

<u>Given:</u> $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$,
$m(\angle B) = 140$,
$m(\angle C) = 150$

<u>Find:</u> $m(\angle A)$

12. Simplify.

(a) $\frac{p}{s} - \frac{p-1}{s-t}$ (b) $\frac{6}{s+4} - \frac{3}{s+1}$

13. If a man owns 2/3 of a piece of property and sells 3/5 of his holding for \$2400, what is the value of the entire property?

14. Find a substitution for 'k' such that the equation '$x^2 - 4x + 2k = 0$' has two rational roots.

15. One kind of steel alloy is 0.19% carbon. How much carbon is contained in 1730 pounds of this steel?

16. A regular pentagon has center (0, 0) and one vertex (3, 0). What are the coordinates of the other vertices?

17. Solve: (a) $2x\sqrt{0.05} = 5$ (b) $1/x = \sqrt{0.16}$

18. Find the measure of an altitude of an equilateral triangle whose side-measure is 40.

19. Find three consecutive integers such that the product of the two smallest differs from the product of the two largest by the sum of the largest and the smallest.

20. Factor.

(a) $x^2 + xy + xz + yz$ (b) $sx^2 + tx^2 + 2s + 2t$

21. If a ball is thrown upwards with a velocity of v feet per second, the height reached at the end of t seconds is s feet, where $s = vt - 16t^2$. If the starting velocity is 100 feet per second, how long will it take the ball to reach its greatest height?

22. Simplify.

(a) $\dfrac{6x^{2n} + 6x^{3n} - 12x^{4n}}{3x^n}$

(b) $\dfrac{x^{1+m} - x^{1+2m} - x^{1+3m} + x^{1+4m}}{x^{3m}}$

23. Suppose that the difference in area-measures of the circumscribed and inscribed squares of a circle is 72. What is the area-measure of the circle?

24. A magic square is a table of numbers having as many rows as columns and such that the numbers listed in each column, each row, and each diagonal have the same sum. Examples:

8	1	6
3	5	7
4	9	2

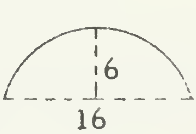
3-square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

4-square

What is the constant sum for a 50-square which lists the first 2500 positive integers?

25. How long is a circular arc whose chord is 16 feet long and whose "height" is 6 feet?



9.02 The need for a new basic principle.--On page 9-3 we stated that the existence condition:

$$(\dagger_1) \qquad \forall_{x \geq 0} \exists_z \, (z \geq 0 \text{ and } z^2 = x),$$

which is needed in order to justify our introduction of the square root operator, is not a consequence of our basic principles. In this section we shall substantiate this statement. The procedure will be to show, first, that all our basic principles continue to hold if they are reinterpreted as statements about only the rational real numbers, and, second, that $(\dagger_1)$ becomes false when it is interpreted as referring only to rational real numbers.

THE SYSTEM OF RATIONAL REAL NUMBERS

Recall that, by definition, a real number is rational if and only if it is the quotient of an integer by a nonzero integer. For example,

$$2/9, \quad 1, \quad \sqrt{100}, \quad 0, \quad -3.125, \quad 0.8\overline{3}, \quad \text{and} \quad \sqrt{8}/\sqrt{2}$$

are rational real numbers. Be sure you understand why this is so [for example, $\sqrt{100} = 10/1$, and 10 and 1 are integers]. Then, give some other examples of rational numbers.

If r and s are rational numbers then there are integers--say, i, j, k, and ℓ [with $j \neq 0$, and $\ell \neq 0$]--such that

$$r = \frac{i}{j} \quad \text{and} \quad s = \frac{k}{\ell}.$$

It follows that

$$(1) \qquad r + s = \frac{i\ell + kj}{j\ell}. \qquad \text{[Theorem 57]}$$

Since i, j, k, and ℓ are integers, $i\ell + kj$ and $j\ell$ are also integers [Theorems 110d and 110b] and, since $j \neq 0$ and $\ell \neq 0$, $j\ell \neq 0$ [Theorem 55]. Hence, by (1) and definition, it follows that r + s is rational. Consequently, for each rational number r and each rational number s, r + s is a rational number. This result can be restated as:

> The set of rational real numbers is closed with respect to addition.

Once you have understood this statement and its proof, you can restate the proof more briefly:

Since the set of integers is closed with respect to multiplication and addition, and since the set of nonzero real numbers is closed with respect to multiplication, it follows from Theorem 57 [and the definition of 'rational number'] that the set of rational numbers is closed with respect to addition.

In just the same way you can prove that the set of rational real numbers is closed with respect to the operations of multiplication, opposition, subtraction, and division [by a nonzero rational number].

These results on closure show that we can consider addition, multiplication, opposition, subtraction, and division as operations defined on the set consisting of just the rational real numbers. Since, also, 0 and 1 are rational, it is easy to see that our first 11 basic principles become true statements about rational numbers if we think of '+', '$\cdot$', '$-$', '-', and '$\div$' as referring to these restricted operations on rational real numbers, and read '$\forall_x$', say, as 'for each rational number x'.

The same holds for the basic principles (P_1) - (P_4) and (G) if we take 'P' to be a name for the set of positive rational numbers and '>' as referring to the greater-than relation for rational numbers. Since integers are rational numbers, the same holds of (I^+_1) - (I^+_3), (C), and (I).

As a consequence of the preceding remarks every theorem which we could derive from our basic principles is, when reinterpreted as above, a true statement about the system of rational numbers. A person looking at our basic principles could not tell whether we are talking about the real numbers or only the rational numbers.

Now, let's consider the statement $(\dagger_1)$:

$$\forall_{x \geq 0} \exists_z \, (z \geq 0 \text{ and } z^2 = x)$$

Can this be derived from our basic principles? If it could, we would be able to show, since 8, say, is a nonnegative rational number, that there is a <u>rational</u> number z such that $z \geq 0$ and $z^2 = 8$. But, on page 4-48 of Unit 4, we proved that there is no such rational number. [In the proof we used only theorems which follow from our present set of basic principles.] A person who is told that $(\dagger_1)$ is a basic principle [or, as it later will be, a theorem] knows that we are not talking about rational numbers.

Consequently, $(\dagger_1)$ is not a consequence of our basic principles, and we do need some additional basic principle to justify our dealing with square roots.

EXPLORATION EXERCISES

A. 1. Consider the set, N, of negative numbers.

(a) Is there a number which is greater than or equal to each member of N?

(b) Is there more than one number z such that, for each $x \in N$, $z \geq x$?

(c) Is there a $z \in N$ such that $\forall_x [x \in N \Rightarrow z \geq x]$?

2. Repeat Exercise 1 for the set, I^+, of positive integers.

3. Consider the set, $N \cup \{0\}$, of nonpositive numbers.

(a) Is there a z such that $\forall_x [x \in N \cup \{0\} \Rightarrow z \geq x]$?

(b) Is there more than one such z?

(c) Is there a $z \in N \cup \{0\}$ such that $\forall_x [x \in N \cup \{0\} \Rightarrow z \geq x]$?

(d) Is there more than one such z?

4. (a) Is there a z such that, for each x, if x belongs to the set of positive rationals less than $\sqrt{2}$ then $z \geq x$? Is there more than one such z?

(b) Is there a z in the set of positive rationals less than $\sqrt{2}$ such that, for each x, if x also belongs to this set then $z \geq x$? Is there more than one such z?

5. Consider the set M, where $M = \{y\colon -5 < y < -4 \text{ or } y = 0 \text{ or } 1 < y \leq 2\}$.

(a) Is there a z such that $\forall_x [x \in M \Rightarrow z \geq x]$? Is there more than one such z?

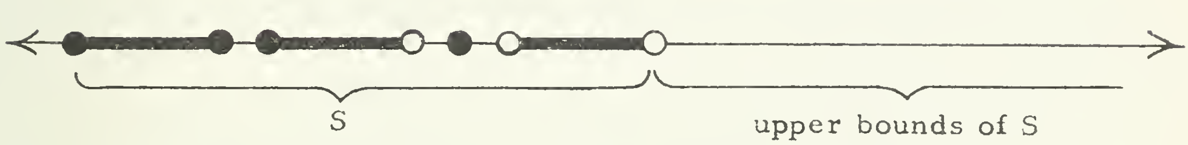
(b) Is there a $z \in M$ such that $\forall_x [x \in M \Rightarrow z \geq x]$? Is there more than one such z?

*

The foregoing exercises deal with the ideas of <u>upper bound</u> and <u>greatest member</u>. Here are definitions:

$$\forall_S \forall_z \; [z \text{ is an upper bound of } S \iff \forall_x \, [x \in S \Rightarrow z \geq x]]$$

$$\forall_S \forall_z \; [z \text{ is a greatest member of } S \iff (z \text{ is an upper bound of } S \text{ and } z \in S)]$$



The set S pictured above has upper bounds but no greatest member. What would be the easiest way to change S so that it would have a greatest member?

*

6. (a) Give two upper bounds for $\{t: 3t + 1 < 10\}$. Does this set have a greatest member?

   (b) Give two upper bounds for $\{t: 3t + 1 \leq 10\}$. Does this set have a greatest member?

7. Let R be the set of reciprocals of the positive integers.

   (a) Does R have an upper bound?

   (b) Does R have a greatest member?

8. Suppose that t is an upper bound of some given set S. What other numbers are sure to be upper bounds of S? Justify your answer.

9. Suppose that, for some given set S, S has a greatest member. Then,

$(\dagger_1) \qquad \exists_z (\forall_x [x \in S \Rightarrow z \geq x] \text{ and } z \in S).$

(a) Check this by referring to the definitions.

(b) State the conditions which says that S cannot have two greatest members:

$(\dagger_2) \qquad \forall_y \forall_z [((\forall_x [x \in S \Rightarrow y \geq x] \text{ and } y \in S)$
$\text{and } (\qquad\qquad \text{and} \qquad))$
$\Rightarrow \qquad = \qquad]$

(c) Prove the uniqueness condition $(\dagger_2)$. [Hint. See Theorem 93.]

(d) Fill in the blanks.

Since, no matter what set of real numbers S is, the ____?____ condition $(\dagger_2)$ is a theorem, it follows that, for any set S such that the ____?____ condition $(\dagger_1)$ is a theorem, we are justified in speaking of

_?_ z such that $(\forall_x [x \in S \Rightarrow z \geq x] \text{ and } z \in S)$

--that is, of

_?_ ____?____ member of S.

10. Complete these definitions of <u>lower bound</u> and <u>least member</u>.

(a) $\forall_S \forall_z [z \text{ is a lower bound of } S \Longleftrightarrow \forall_x [x \in S \Rightarrow z \qquad x]]$

(b) $\forall_S \forall_z [z \text{ is a least member of } S \Longleftrightarrow$
$(\qquad\qquad \text{and} \qquad)]$

11. Fill in.

(a) Just as in Exercise 9, one can prove that no set can have two least members. Hence, if a set has a least member, one is justified in calling it _?_ ___?___ ____?____ of S.

(b) Suppose that U is the set of upper bounds of some set S. If U has a least member then this member of U is called _?_ ___?___ upper bound of _?_.

12. Fill in.

(a) If U is the set of upper bounds of some set S then U is either a right half-line or a right ___?___ or the whole ___?___.

(b) If U is the set of upper bounds of some set S then each member of S is a ___?___ bound of U.

(c) Suppose that U is the set of upper bounds of some set S. If U has a least member then this member of U is called __?__ ___?___ upper bound of __?__.

13. Name a set which has an upper bound but does not have a least upper bound. [Hint. There is only one such set.]

14. Give an example of a set of real numbers which has

(a) both an upper bound and a lower bound

(b) a lower bound but no upper bound

(c) an upper bound but no lower bound

(d) neither an upper nor a lower bound

(e) a greatest member but no upper bound

(f) an upper bound which belongs to it

(g) a least upper bound which does not belong to it

(h) a lower bound but not a greatest lower bound

15. Which of the following sets has 3 as an upper bound?

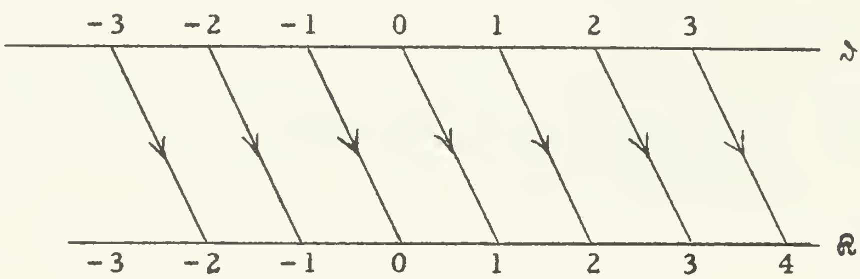
(a) $\{x: x < 3\}$ (b) $\{x \geq 2: x < 3\}$

(c) $\{x \geq 3: x < 3\}$ (d) $\{x \geq 4: x < 3\}$

(e) $\{x \geq 3: x \leq 3\}$ (f) $\{x: x \leq 3\}$

16. Which of the sets listed in Exercise 15 has 3 as its least upper bound?

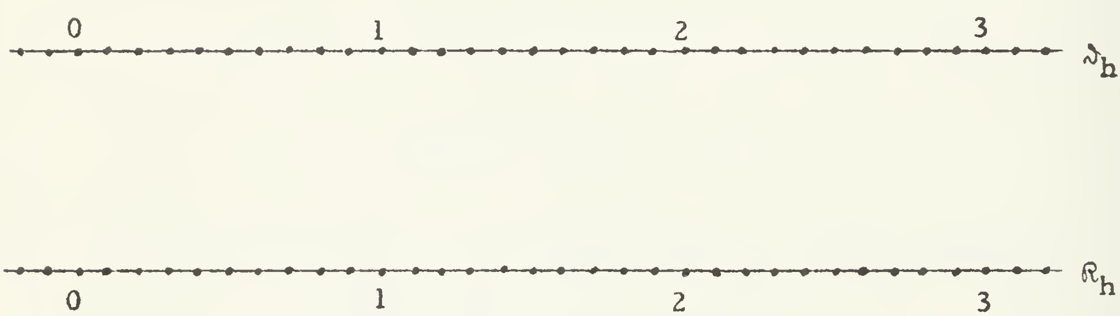
B. In Unit 5 you learned to picture a function [i. e. a mapping] by picturing its domain and range and drawing arrows. For Example, the adding 1 function "looks" like this:



1. Make a similar picture of the function h such that

$$h(x) = \frac{2(x + 1)}{x + 2}, \quad \text{for } x \geq 0.$$

(a) At first consider only the arguments 0, 0.5, 1.1, and 1.2. [Carry out your computations correct to two decimal places, for if you don't, you may miss the point.]



(b) Your work in part (a) may suggest that, for any $a \geq 0$, $h(a) > a$. How does this show up in your picture?

(c) To check the suggestion of part (b), here is a table listing more values of the function [correct to four decimal places].

a	1.3	1.4	1.41	1.414
h(a)	1.3939	1.4118	1.4135	1.4141

Do these values tend to confirm the suggestion that, for any $a \geq 0$, $h(a) > a$?

(d) You can check the conjecture [that, for any $a \geq 0$, $h(a) > a$] more completely by carrying out the following manipulations:

$$h(a) - a = \frac{2(a+1)}{a+2} - a = \frac{?}{a+2}$$

Do these manipulations, obtaining as simple a result as you can, and infer a theorem:

$$\forall_{x \geq 0}\ h(x) - x = \frac{?}{x+2}$$

(e) Does the theorem you proved in part (d) verify the conjecture? If not, what less sweeping conjecture does it verify?

2. You have seen in Exercise 1 that

(Φ_1) $\forall_{x \geq 0}[x^2 < 2 \Rightarrow h(x) > x]$ and $\forall_{x \geq 0}[x^2 > 2 \Rightarrow h(x) \overset{?}{\downarrow} x]$.

[Complete this last.]

(a) Check this second generalization by drawing more arrows in the picture you drew for Exercise 1(a). Use arguments whose squares are greater than 2, such as those listed in the following table.

a	1.5	1.7	2	3
h(a)	1.428	1.459	1.5	1.6

(b) A more careful look at the figure you made in answering Exercise 1(a) may suggest another conjecture. Notice that the arrows corresponding to arguments of h whose squares are very close to 2 are very nearly vertical. This means that the value of the function, for such an argument, differs very little from the argument itself. This might suggest that

(Φ_2) $\forall_{x \geq 0}[x^2 < 2 \Rightarrow [h(x)]^2 < 2]$ and $\forall_{x \geq 0}[x^2 > 2 \Rightarrow [h(x)]^2 > 2]$.

You can check these conjectures much as you did the conjecture in Exercise 1(d) by considering the expression:

$$2 - [h(a)]^2$$

Do so, thus proving the conjectures.

C. Consider the set S such that $S = \{x \geq 0: x^2 < 2\}$.

1. Show that S has no greatest member. [*Hint*. You can do this by showing that, for any number in S, there is a larger number which is also in S. See the first generalizations in (Φ_1) and (Φ_2) of Exercise 2 of Part B.]

2. Suppose that b is a nonnegative number whose square is 2. Prove:

   (a) b is an upper bound of S. [*Hint*. Use Theorem 98b.]

   (b) $\forall_{x \geq 0} [x < b \Rightarrow x \in S]$ [*Hint*. Use Theorem 98c.]

   (c) No number less than b is an upper bound of S. [*Hint*. Use Exercises 1 and 2(b).]

   (d) b is the *least* upper bound of S.

9.03 *The least upper bound principle*. --In Part C of the preceding exercises you considered the set S such that $S = \{x \geq 0: x^2 < 2\}$. You proved that

> *if* there is a nonnegative number whose square is 2
> *then* S has a least upper bound.

You did this by showing that any nonnegative number whose square is 2 is a least upper bound of S.

Let's consider the converse problem. Is it the case that

> *if* S has a least upper bound
> *then* there is a nonnegative number whose square is 2?

As we shall see, the answer is 'yes'. We shall prove this, as you might suspect, by showing that any least upper bound of S is a nonnegative number whose square is 2. In doing so, we shall use the function h which you studied in Part B on pages 9-28 and 9-29.

Recall that $$h(x) = \frac{2(x + 1)}{x + 2}, \text{ for } x \geq 0,$$

and that you proved the generalizations:

(Φ_1) $\forall_{x \geq 0}[x^2 < 2 \Rightarrow h(x) > x]$ and $\forall_{x \geq 0}[x^2 > 2 \Rightarrow h(x) < x]$

(Φ_2) $\forall_{x \geq 0}[x^2 < 2 \Rightarrow [h(x)]^2 < 2]$ and $\forall_{x \geq 0}[x^2 > 2 \Rightarrow [h(x)]^2 > 2]$

Suppose, then, that b is a least upper bound of S. We wish to show that b is a nonnegative number whose square is 2. That $b \geq 0$ is obvious since $0 \in S$ and b is an upper bound of S. To show that $b^2 = 2$ we shall show that $b^2 \not< 2$ and $b^2 \not> 2$. Then, it will follow [by Theorem 86a] that $b^2 = 2$.

So, suppose, first, that $b^2 < 2$. Since $b \geq 0$ and $b^2 < 2$, it follows, from (Φ_1), that $h(b) > b$ and, from (Φ_2), that $[h(b)]^2 < 2$. So, since $h(b) \geq 0$, it follows that h(b) is a member of S which is greater than b. But, this is impossible because b is an upper bound of S. Hence, $b^2 \not< 2$.

Suppose, now, that $b^2 > 2$. Since $b \geq 0$ and $b^2 > 2$, it follows, from (Φ_1), that $h(b) < b$ and, from (Φ_2), that $[h(b)]^2 > 2$. Also, since $b \geq 0$, it follows from the definition of h that $h(b) \geq 0$. Now, since $[h(b)]^2 > 2$, $[h(b)]^2$ is greater than the square of any member of S. So, [by Theorem 98b], since $h(b) \geq 0$, h(b) is an upper bound of S. But, since $h(b) < b$, and since b is the least upper bound of S, this is impossible. Hence, $b^2 \not> 2$.

Since $b^2 \not< 2$ and $b^2 \not> 2$, it follows that $b^2 = 2$.

Combining what we have just proved with what you proved in Part C, we see that the statements:

$$\exists_{z \geq 0}\ z^2 = 2$$

and:

$$\{x \geq 0:\ x^2 < 2\} \text{ has a least upper bound}$$

are equivalent statements. More generally, the statements:

(1) $\forall_{y > 0} \exists_{z \geq 0}\ z^2 = y$

and:

(2) $\forall_{y > 0}\ \{x \geq 0:\ x^2 < y\}$ has a least upper bound

are equivalent.

[Are '$\exists_{z \geq 0}\ z^2 = 0$' and '$\{x \geq 0:\ x^2 < 0\}$ has a least upper bound' equivalent? Explain.]

As was pointed out on page 9-4, our work with square roots could be justified by adopting one new basic principle:

$$(3) \qquad \forall_{y \geq 0} \exists_{z \geq 0}\, z^2 = y \qquad \text{[See } (\dagger_1) \text{ on page 9-22.]}$$

Since we can already prove that $0^2 = 0$, we could also get by [as far as square roots are concerned] by adopting (1) on page 9-31 as a new basic principle. And, since (1) and (2) are equivalent, we could, instead, adopt (2). Now, by itself, (2) seems to have no advantage over (1). However, we also wish to be able to deal with cube roots [see Part K on page 9-12] and, as we shall see, with the inverses of many other functions besides those of $\{(x, y), x \geq 0: y = x^2\}$ and $\{(x, y): y = x^3\}$. To do so we could, of course, adopt a basic principle like (1) for each such function. This would, obviously, be inelegant. The advantage of (2) is that, unlike (1), it suggests a basic principle which, as we shall see, covers all the cases we shall wish to consider. This new basic principle--the least upper bound principle [ℓubp]--is:

Each nonempty set which has an upper bound has a least upper bound.

With this additional principle, it is easy to see that (2), for example, is a theorem. [Just show that, for $b > 0$, $\{x \geq 0: x^2 < b\}$ is nonempty and that it has an upper bound.] Consequently, (3) is also a theorem, and our work with square roots rests on a firm foundation.

EXERCISE

Use the ℓubp to show that I^+ does not have an upper bound. [Hint. Since $1 \in I^+$, I^+ is nonempty. So, if I^+ has an upper bound then, by the ℓubp, I^+ has a least upper bound--say, b. Since $b - 1 < b$, it follows that $b - 1$ is not an upper bound of I^+--that is, it follows that there is a positive integer p such that . . . . [Complete the proof by using (I_2^+) to obtain a contradiction.]

* * *

In Unit 7 we adopted the cofinality principle:

$$\text{(C)} \qquad \forall_x \exists_n\, n > x$$

as a basic principle. Now what (C) says is just that it is not the case that there is a number x such that each positive integer is less than or equal to x [Explain.]. That is, (C) says that I^+ does not have an upper bound. Since, as you have seen, this follows from the ℓubp, (I^+_1), (I^+_2) and some [three] earlier theorems which do not depend on (C), we need no longer take (C) as a basic principle. With the adoption of the ℓubp, (C) has become a theorem.

Moreover, it is possible to show that [except for definitions, such as (G) and (I), and recursive definitions] our first eleven basic principles together with (P_1) - (P_4), (I^+_1) - (I^+_3), and the ℓubp, form a complete set of basic principles for the system of real numbers. No other basic principles will ever be needed [except definitions and defining principles like $(*_1)$ on page 9-5].

* * *

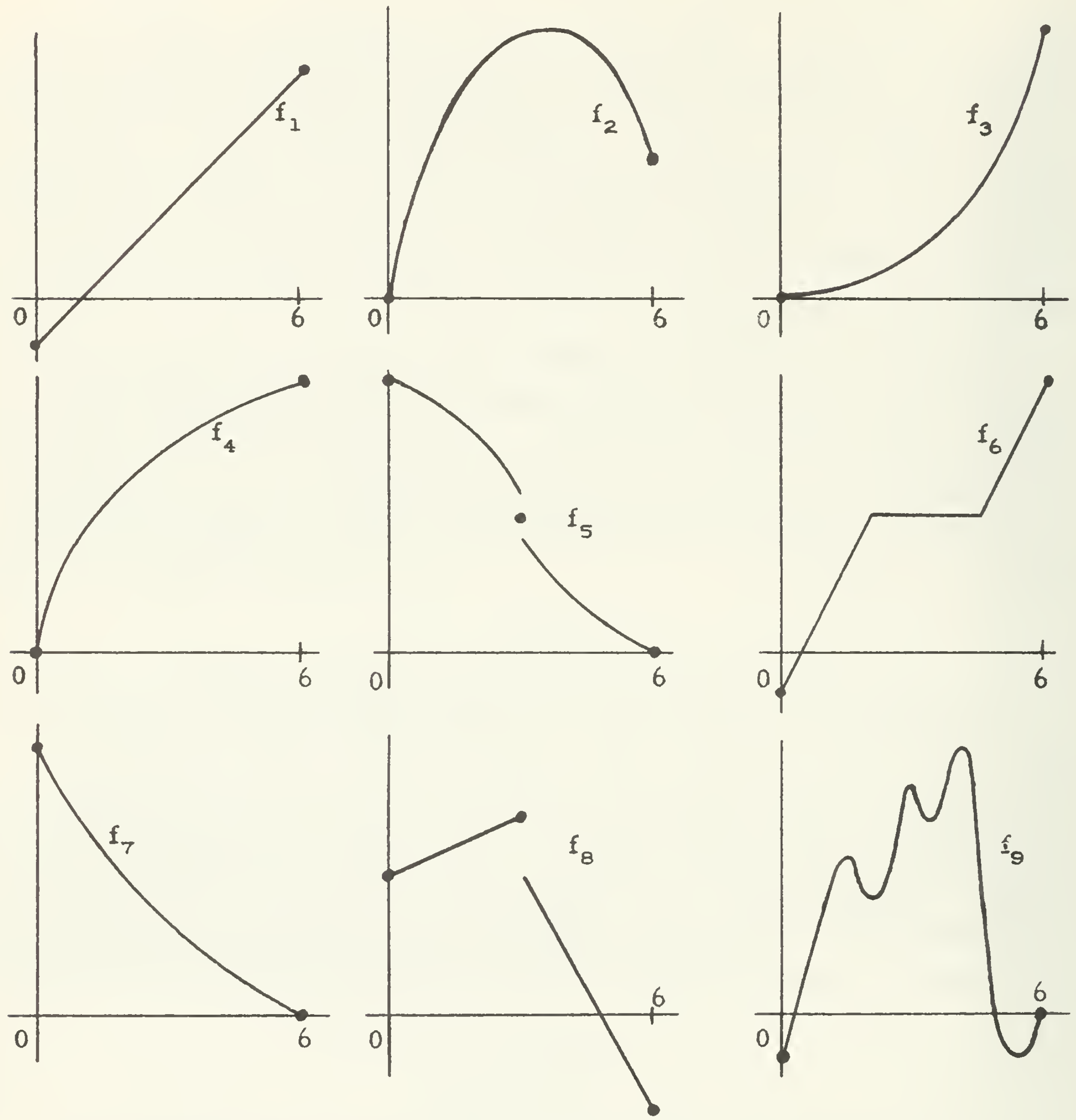
In the Exploration Exercises beginning on page 9-34, we shall develop some general concepts one of which will give a new insight into the role of the ℓubp, and we shall state but not prove a theorem about inverse functions which will make it easy to settle such questions as the existence of square roots, cube roots, etc. [In your later study of mathematics, there will be other questions of this kind, and the theorem just referred to will enable you to settle each of them very easily.]

* * *

For a more complete treatment of the topics covered in the following Exploration Exercises and in the introductory paragraph of section 9.04, you may wish to study Appendix A [pages 9-190 through 9-230]. If you do this, you should take up the text again with the subsection RADICAL EXPRESSIONS on page 9-50.

EXPLORATION EXERCISES

A. Here are graphs of several functions, each having the segment $\overline{0, 6}$ as domain.



1. Which of these functions do not have inverses? [Recall that a function f has an inverse if and only if

$$\forall_{x_1 \in \mathcal{D}_f} \forall_{x_2 \in \mathcal{D}_f} [f(x_1) = f(x_2) \Rightarrow x_1 = x_2]. \qquad]$$

*

Suppose that you pick two arguments of a function. If it turns out that, however you do this, the function value for the larger argument is larger than the value for the smaller argument, then the function is said

to be <u>increasing</u>. In other words, a function f is increasing if and only if

$$\forall_{x_1 \in \mathscr{D}_f} \forall_{x_2 \in \mathscr{D}_f} [x_2 > x_1 \Rightarrow f(x_2) > f(x_1)].$$

*

2. Which of the nine functions pictured above are increasing?

3. Record your answers to Exercises 1 and 2 in the following table.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
No inverse		✓							
Increasing	✓								
Decreasing									

4. Complete:

A function f is decreasing
if and only if

$$\forall_{x_1 \in \mathscr{D}_f} \forall_{x_2 \in \mathscr{D}_f} [x_2 > x_1 \Rightarrow \qquad\qquad\qquad].$$

5. Complete the third line of the table in Exercise 3.

6. If you have done the foregoing exercises correctly, you will find that no column in the table of Exercise 3 contains two checkmarks. This illustrates the fact that [pick the right words]

if a function $\left\{\begin{matrix}\text{does}\\ \text{does not}\end{matrix}\right\}$ have an inverse

then the function is $\left\{\begin{matrix}\text{either}\\ \text{neither}\end{matrix}\right\}$ increasing $\left\{\begin{matrix}\text{or}\\ \text{nor}\end{matrix}\right\}$ decreasing

7. If you did Exercise 6 correctly, you probably discovered that

if a function is either increasing or decreasing
then the function has an inverse.

Which of the functions pictured on page 9-34 is a counterexample to the <u>converse</u> of this statement?

8. Sketch a graph of an increasing function, and on the same chart, sketch a graph of its inverse. Is the inverse increasing, decreasing, or neither?

9. Is there an increasing function whose inverse is not increasing?

10. (a) Consider a function that has an inverse. Do any of its subsets have inverses?

    (b) Consider a function that does not have an inverse. Do any of its subsets have inverses?

    (c) Consider a function that is decreasing. Are any of its subsets increasing?

B. For each n, the function $\{(x, y): y = x^n\}$ is called _the nth power function_. [The function $\{(x, y), x \geq 0: y = x^n\}$ is the nth power function restricted to nonnegative arguments.]

1. Sketch, on separate charts, graphs of the 1st, 2nd, 3rd, and 4th power functions for arguments between -4 and 4. [You may find it convenient to use different scales for the horizontal and vertical axes.]

2. Sketch, on separate charts, graphs of the converses of the first four power functions.

3. Which of the first four power functions have inverses? Which of all the positive-integral power functions have inverses?

4. Which of the positive-integral power functions are increasing?

5. Do your answers to Exercises 3 and 4 support your finding in Exercise 6 of Part A?

6. Consider the positive-integral power functions restricted to nonnegative arguments.

   (a) Which are increasing? (b) Which have increasing inverses?

* * *

In Part A of the preceding exercises you discovered that any function which is increasing or decreasing has an inverse of the same type. It is convenient to introduce a word to cover both kinds of functions:

> Definition.
>
> A function is monotonic if and only if it is either increasing or decreasing.

So, the theorem you discovered is:

> Theorem 184.
>
> Each monotonic function has a monotonic inverse of the same type.

In Part B you discovered that the positive-integral power functions restricted to nonnegative arguments are increasing--that is:

> Theorem 185.
>
> $$\forall_n \forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_2 > x_1 \Rightarrow x_2^n > x_1^n]$$

You also discovered that the odd positive-integral power functions are increasing [all the way!]--that is:

> Theorem 185′.
>
> $$\forall_n \forall_{x_1} \forall_{x_2} [x_2 > x_1 \Rightarrow x_2^{2n-1} > x_1^{2n-1}]$$

Applying Theorem 184 to the functions mentioned in Theorems 185 and 185′ we see that

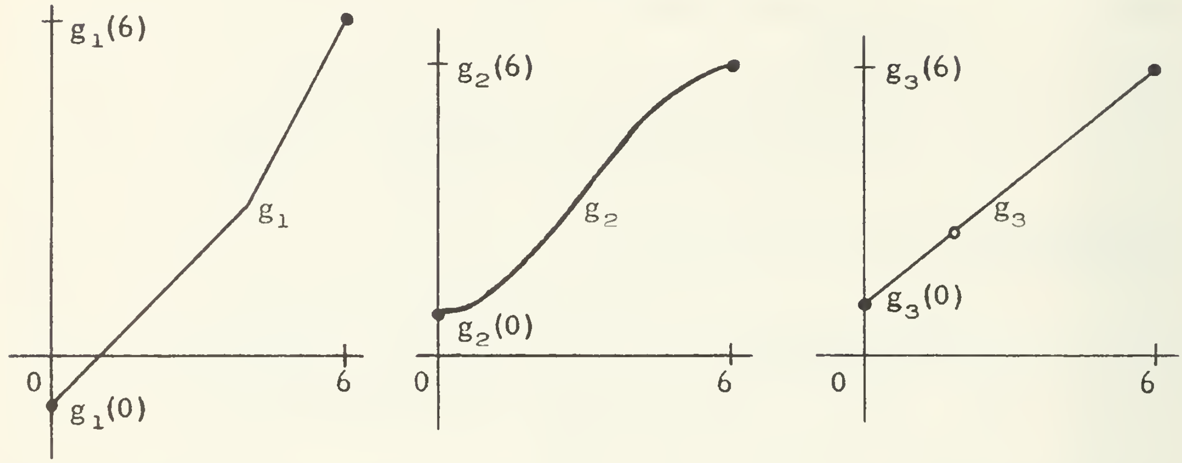
(a) each restricted positive-integral power function has an increasing inverse, and

(b) each odd positive-integral power function has an increasing inverse.

[Proofs of Theorems 184 and 185 are given in Appendix A. Theorem 185′ is discussed on page 9-54.]

* * *

C. Here are graphs of several increasing functions each of which has some subset of the segment $\overline{0, 6}$ as its domain.



1. The domain of all but one of the three functions pictured above is the entire segment $\overline{0, 6}$. Which one?

2. Look at the graph of g_1. Since g_1 is an increasing function, its range is bound to be a subset of the segment $\overline{g_1(0), g_1(6)}$. Explain.

3. From the graph, the range of g_1 is the entire segment $\overline{g_1(0), g_1(6)}$. For which of the other two functions can you make similar statements?

4. Record the results you obtained in Exercises 1 and 3 in the following table.

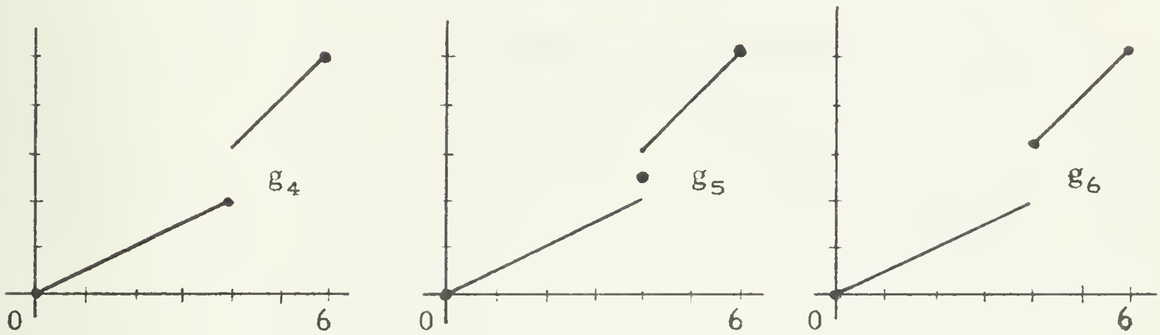
n	1	2	3
$\mathcal{R}_{g_n} \neq \overline{g_n(0), g_n(6)}$			
$\mathcal{D}_{g_n} = \overline{0, 6}$	✓		

D. The results recorded in the table of Exercise 4 of Part C might suggest that, for any increasing function g, the range of g is an entire segment if and only if the domain of g is an entire segment. Let's investigate this conjecture by considering some additional functions

whose domain is the entire segment $\overline{0, 6}$.

$$g_4(x) = \begin{cases} x/2, & 0 \le x \le 4 \\ x - 1, & 4 < x \le 6 \end{cases} \qquad g_5(x) = \begin{cases} x/2, & 0 \le x < 4 \\ 2.3, & x = 4 \\ x - 1, & 4 < x \le 6 \end{cases}$$

$$g_6(x) = \begin{cases} x/2, & 0 \le x < 4 \\ x - 1, & 4 \le x \le 6 \end{cases}$$



1. (a) What is $g_4(0)$? $g_4(6)$?

   (b) The shortest <u>segment</u> which contains the range of g_4 is $\overline{?, ?}$.

   (c) What is the shortest segment which contains $\mathcal{R}_{g_5}$? $\mathcal{R}_{g_6}$?

2. (a) Is 3.98 an argument of g_4? If so, what is $g_4(3.98)$?

   (b) Is 4 an argument of g_4? If so, what is $g_4(4)$?

   (c) Is 4.01 an argument of g_4? If so, what is $g_4(4.01)$?

3. (a) Is 2 a value of g_4--that is, does there exist an $x \in \mathcal{D}_{g_4}$ such that $g_4(x) = 2$?

   (b) True or false?

   (1) $\exists_{x \in \mathcal{D}_{g_4}} \; g_4(x) = 2.01$ (2) $\mathcal{R}_{g_4} = \overline{g_4(0), g_4(6)}$

4. Complete.

   (a) $\mathcal{R}_{g_4} = \{y\colon 0 \le y \quad 2 \text{ or } 3 \quad y \le 5\}$ ['<' or '≤']

   (b) $\mathcal{R}_{g_6} = \{y\colon \qquad \text{or} \qquad \}$

   (c) $\mathcal{R}_{g_5} = \{y\colon \qquad \text{or} \qquad \text{or} \qquad \}$

* * *

In Exercise 3 of Part D you found that, although g_4 is an increasing function whose domain is the entire segment $\overline{0, 6}$, the range of g_4 is not the entire segment $\overline{g_4(0), g_4(6)}$. So, the conjecture that the range of an increasing function is a segment if and only if its domain is a segment turns out to be false. [The conjecture has two parts--an if-part and an only if-part. For which part is g_4 a counterexample?]

We wish to find a sufficient condition that an increasing function whose domain is a segment have a segment as its range. To do so, let's see how g_4 [and g_5 and g_6] differ from the functions g_1, g_2, and g_3 which led us to our false conjecture. Your answers for Exercise 2 of Part D may suggest such a difference. You will explore this further in Part E.

* * *

E. 1. Complete the following table for g_4.

x	3.9	3.98	3.998	4	4.01	4.001	4.0001	4.00001
$g_4(x)$		1.99		2	3.01			
$\lvert g_4(x) - g_4(4)\rvert$		0.01		0	1.01			

2. (a) Complete.

You can find an $x \in \mathcal{D}_{g_4}$ as close as you wish to the argument 4 and such that $|g_4(x) - g_4(4)| \geq$ ___ .

(b) True or false?

There is a number $c > 0$ such that no matter how close you come to the argument 4 of g_4 you can find an $x \in \mathcal{D}_{g_4}$ such that $|g_4(x) - g_4(4)| \geq c$.

3. Look at the graph of the function g_1 of Part C on page 9-38. The function g_1 is defined by:

$$g_1(x) = \begin{cases} x - 1, & 0 \leq x \leq 4 \\ 2x - 5, & 4 < x \leq 6 \end{cases}$$

Complete the table at the top of the next page.

x	4	3.9	3.99	3.998	4.05	4.005	4.001	4.0001
$g_1(x)$								
$\lvert g_1(x) - g_1(4)\rvert$								

4. (a) Explain.

   No matter what $x \in \mathcal{D}_{g_1}$ you choose, as long as it is close enough to the argument 4 you can be sure that $|g_1(x) - g_1(4)| < 0.1$.

   (b) In part (a), how close need the argument x be to 4?

5. Complete.

   (a) No matter what $x \in \mathcal{D}_{g_1}$ you choose, as long as x differs from 4 [either way] by less than _____, you can be sure that $|g_1(x) - g_1(4)| < 0.01$.

   (b) No matter what $x \in \mathcal{D}_{g_1}$ you choose, as long as $|x - 4| <$ _____ you can be sure that $|g_1(x) - g_1(4)| < 0.002$.

   (c) No matter what $x \in \mathcal{D}_{g_1}$ you choose, as long as $|x - 4| <$ _____ you can be sure that $|g_1(x) - g_1(4)| < 0.0002$.

6. (a) Explain.

   No matter what number $c > 0$ you choose [however small], if x is any argument of g_1 which is sufficiently close to 4 then $|g_1(x) - g_1(4)| < c$.

   (b) In part (a), how close need the argument x be to 4?

7. True or false?

   (a) Choose any $x_0 \in \mathcal{D}_{g_1}$. No matter what number $c > 0$ you choose, if x is any argument of g_1 which is sufficiently close to x_0 then $|g_1(x) - g_1(x_0)| < c$.

   (b) Choose any $x_0 \in \mathcal{D}_{g_1}$ and any $c > 0$. You can't find arguments x of g_1 arbitrarily close to x_0 for which $|g_1(x) - g_1(x_0)| \geq c$.

* * *

Each of the functions g_1 and g_4 is an increasing function whose domain is a segment--the segment $\overline{0, 6}$. The range of g_1 is the entire segment $\overline{g_1(0), g_1(6)}$, but the range of g_4 is only a part of the segment $\overline{g_4(0), g_4(6)}$. The reason for this difference between g_4 and g_1 is indicated in the exercises of Part E [Exercises 2 and 7].

In Exercise 2 you found that there are arbitrarily small changes in the argument of g_4 which result in a change of at least 1 in the value of g_4.

In Exercise 7 you found that this was not the case with g_1. You can be sure that the value of g_1 will change as little as you wish, if you take care to change its argument little enough--a small change in the argument of g_1 will result in a small change in the value of g_1 and, indeed, sufficiently small changes in the argument will result in arbitrarily small changes in the value.

Functions which share this property with g_1 are said to be continuous at each argument. In contrast, g_4 is discontinuous at its argument 4.

> **Definition.**
>
> A function f is continuous at x_0 if and only if $x_0 \in \mathcal{D}_f$ and $f(x)$ differs arbitrarily little from $f(x_0)$ for each $x \in \mathcal{D}_f$ which is sufficiently close to x_0.
>
> A function is continuous if and only if it is continuous at each of its arguments.

[A more formal definition is given on page 9-211, in Appendix A, preceded by additional Exploration Exercises.]

* * *

F. 1. Look at the graphs of the functions f_1 through f_9 of Part A on page 9-34. Which of these functions do you think are continuous?

2. (a) Which of the positive-integral power functions of Part B [page 9-36] do you think are continuous?

(b) Which of the positive-integral power functions restricted to nonnegative arguments do you think are continuous?

3. (a) Suppose that a function f is continuous at one of its arguments x_o and suppose that f_o is a subset of f which also has x_o as an argument. Is f_o also continuous at x_o? [Hint. To answer this you need to consider the values of f_o at x_o and at nearby arguments of f_o. Think of f_o as what is left when you remove some ordered pairs from f.]

   (b) True or false?

   Each subset of a continuous function is continuous.

   (c) Is the function g_3 of Part C on page 9-38 continuous?

G. 1. Complete the following table, referring to the increasing functions of Parts C and D on pages 9-38 and 9-39.

n	1	2	3	4	5	6
$\mathcal{R}_{g_n} \neq \overline{g_n(0), g_n(6)}$			✓			
$\mathcal{D}_{g_n} = \overline{0, 6}$	✓	✓				
g_n is continuous	✓					

2. If you have done Exercise 1 correctly, you will find that each column contains exactly two checkmarks. In particular, if some column has a checkmark in the middle row then the column contains just one other checkmark. This illustrates the fact that [pick the right words]

   if g is an increasing function whose domain $\left[\begin{matrix}\text{is}\\ \text{is not}\end{matrix}\right]$ a segment $\overline{a, b}$ then $\mathcal{R}_g \left[\begin{matrix}=\\ \neq\end{matrix}\right] \overline{g(a), g(b)}$ if and only if g is continuous.

3. Complete: if g is a monotonic function whose domain is a segment $\overline{a, b}$ then $\mathcal{R}_g$ ______ if and only if g is ______.

4. (a) One of the functions of Part A is a discontinuous function whose domain and range are both entire segments. Which one?

   (b) Can you sketch a graph of an <u>increasing</u> discontinuous function whose domain and range are both entire segments?

* * *

In Part E of the foregoing exercises you discovered the difference between continuous and discontinuous functions. In Part F you probably guessed that the positive-integral power functions are continuous--for any n, if $|x_2 - x_1|$ is small enough then $|x_2^n - x_1^n|$ will be as small as you wish. This guess is correct:

> Theorem 186.
>
> Each positive-integral power function is continuous.

[Theorem 186 is proved in Appendix A.] Since [Exercise 3 of Part F] each subset of a continuous function is continuous, it follows that each of the positive-integral power functions restricted to nonnegative arguments is continuous.

You discovered in Part G that, for any monotonic function g whose domain is a segment $\overline{a, b}$,

(*) $\mathcal{R}_g = \overline{g(a), g(b)}$ if and only if g is continuous.

Theorems 185, 186, and the if-part of (*) can be used to give another proof [besides the one given in section 9.03] that there is a nonnegative number whose square is 2. Briefly, by Theorems 185 and 186, the function

$$\{(x, y),\ x \geq 0:\ y = x^2\}$$

is an increasing continuous function. So, the same is true of its subset

$$\{(x, y),\ 0 \leq x \leq 3:\ y = x^2\}.$$

This latter function has the additional property that its domain is the segment $\overline{0, 3}$. So, by the if-part of (*), the range of this subset is the segment $\overline{0^2, 3^2}$. So, since $0 \leq 2 \leq 9$, 2 belongs to the range of this function--that is, there is an x such that $0 \leq x \leq 3$ and $2 = x^2$.

As you know from section 9.02, this result must depend, somehow, on the least upper bound principle. In fact, the proof of the if-part of (*) which is given in Appendix A is very similar to the proof in section 9.03 that the least upper bound of $\{x \geq 0:\ x^2 < 2\}$ is a nonnegative number whose square is 2. The ℓubp is used in the same way both in this

proof and in the proof of the if-part of (*).

Using Theorem 184 and both parts of (*) one can prove a fundamental theorem about inverses which we shall need at several critical points in this and later units. The theorem is:

> **Theorem 187.**
> Each continuous monotonic function f whose domain is a segment $\overleftrightarrow{a, b}$ has a continuous monotonic inverse of the same type whose domain is the segment $\overleftrightarrow{f(a), f(b)}$.

The proof, also given in Appendix A, is very simple. The only tricky part is using the only if-part of (*) to show that the inverse of f is continuous. Theorems 185, 186, and 187 can be used, just as we used Theorems 185, 186, and the if-part of (*) to prove that there is a nonnegative number whose square is 2--or, for that matter, to prove that, given any nonnegative number, there is a nonnegative number whose square is the given number.

MISCELLANEOUS EXERCISES

1. Solve these systems.

(a) $\begin{cases} a + 2b + 2c = 11 \\ 2a + b + c = 7 \\ 3a + 4b + c = 14 \end{cases}$ (b) $\begin{cases} x^{-1} + 2y^{-1} - 3z^{-1} = 1 \\ 5x^{-1} + 4y^{-1} + 6z^{-1} = 24 \\ 7x^{-1} - 8y^{-1} + 9z^{-1} = 14 \end{cases}$

2. Prove: $\forall_x \forall_y\ (x^2 + y^2)(x^4 + y^4) \geq (x^3 + y^3)^2$

3. Factor.

(a) $(a + b)^2 - 4c^2$ (b) $t^2 - (5p - 3q)^2$

(c) $1 - (3a - 5b)^2$ (d) $(a - 7b + c)^2 - (7b - c)^2$

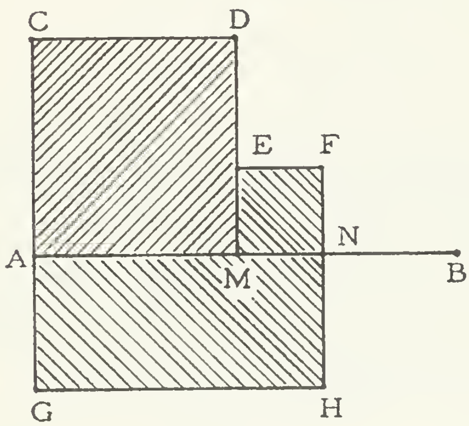
(e) $t^2 - s^2 - 2sr - r^2$ (f) $k^2 - m^2 + 2mx - x^2$

(g) $y^2 - 4by + 4b^2 - a^2 + 2ax - x^2$ (h) $t^3 - s^3$

(i) $1 + d^3$ (j) $8t^3 + 27s^3$

4. Suppose that 5 less than twice a number is greater than 25, and 7 less than three times the number is less than 13 more than twice the number. If the number is prime, what is it?

5. Bill doesn't have enough money to buy a $14 baseball glove. But, if he borrows a third as much money as he now has, he could buy the glove and have more money left than he now lacks. How much money does he now have?

6. 

Hypothesis: ACDM and MEFN are squares,

AM = MB,

AGHN is a rectangle,

NB = NH

Conclusion: K(□ACDM) = K(□MEFN) + K(▭AGHN)

7. Prove.

(a) $\forall_x \forall_y\ (x^2 + xy + y^2)^2 - (x^2 - xy + y^2)^2 = 4xy(x^2 + y^2)$

(b) $\forall_a \forall_b \forall_x \forall_y\ [(ax + by)^2 + (ay - bx)^2][(ax + by)^2 - (ay + bx)^2]$
$= (a^4 - b^4)(x^4 - y^4)$

8. Consider the four triangles into which the diagonals of a convex quadrilateral divide it. Prove that the centroids of these triangles are the vertices of a parallelogram.

9. Factor.

(a) $6ax - 6a^3x^2$

(b) $25 + 15y^2$

(c) $a^2 + ab + ac + bc$

(d) $x^2 + 3x + xy + 3y$

(e) $p^3 - p^2 + p - 1$

(f) $x^2 + 3x + 2$

(g) $r^2 + 54r + 729$

(h) $t^2 + 5tn + 6n^2$

(i) $a^2 - 32ab - 105b^2$

(j) $m^2 + 16m - 260$

10. Derive a formula for the number, C, of square feet of carpeting needed for a rectangular room p feet by q feet if a border r inches wide is to be left between the edge of the carpet and the walls.

11. Consider two circular cylindrical jars of water, the first of which has diameter t times that of the second. If p marbles of the same size are dropped in the first jar, the water level rises a distance a; if q marbles of another size are dropped in the second jar, the water level rises a distance b. Compute the ratio of the diameter of the marbles dropped in the first jar to that of the marbles dropped in the second jar.

12. Simplify.

(a) $\dfrac{x^4 - 9x^2 + 6x - 1}{x^2 - 3x + 1}$ (b) $\dfrac{6x^4 - x^3 - 13x^2 + 5x + 3}{3x^2 - 2x - 1}$

13. If $XY = 3(X - 3)$ and $YZ = 3(Y - 3)$, show that $ZX = 3(Z - 3)$.

14. Solve these equations.

(a) $3(x - 7) + 5(x - 4) = 15$ (b) $3(3 + x) - 3(2x - 5) = 6 - x - 2(3 - x)$

15. If one side of a rectangle is 3 times as long as an adjacent side, how many times as long as the shorter side is the diagonal?

16. Suppose that a, b, and c are sequences such that $a_n = 120 + 2.5n$, $b_n = 90 + 4n$, and $c_n = 85 + 7.5n$. Find the smallest number p such that, for all $q \geq p$, $a_q < b_q < c_q$.

17. A storeclerk uses as a yard-measure a stick which is actually y inches longer than a yard. If he uses this stick in selling an entire bolt of cloth which contains c yards of cloth at s cents per yard, how much has he lost by using the long measure?

18. Simplify.

(a) $\dfrac{1}{x + y} + \dfrac{1}{x - y} - \dfrac{2x}{x^2 + 2xy + y^2}$ (b) $\dfrac{a}{a + b} + \dfrac{b}{a - b} + \dfrac{2ab}{a^2 - b^2}$

9.04 Principal roots.--We have seen that the justification of our work with the principal square root operator lies in two theorems about the squaring function.

(t_1) $\forall_{x \geq 0} \exists_z (z \geq 0 \text{ and } z^2 = x)$

and:

(t_2) $\forall_{x \geq 0} \forall_y \forall_z [((y \geq 0 \text{ and } y^2 = x) \text{ and } (z \geq 0 \text{ and } z^2 = x)) \Rightarrow y = z]$

The first of these, (t_1), can be abbreviated to:

(1) $\forall_{y \geq 0} \exists_{x \geq 0}\ x^2 = y$ ['y' for 'x'; 'x' for 'z']

and the second, (t_2), is equivalent to:

(2) $\forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_1^2 = x_2^2 \Rightarrow x_1 = x_2]$ ['x_1' for 'y'; 'x_2' for 'z']

Looking at (1) and (2), we see $[x \geq 0,\ x_1 \geq 0,\ x_2 \geq 0]$ that these are really statements about the squaring function restricted to nonnegative arguments. What they state is that [(1)] the range of this function contains all nonnegative real numbers and [(2)] this function has an inverse. Since all squares of nonnegative numbers are nonnegative, we may conclude that the restricted squaring function has an inverse whose domain is precisely the set of nonnegative numbers.

You have seen in section 9.02 that the proof of (1) requires a new basic principle--the least upper bound principle--and you have seen at the end of the preceding Exploration Exercises how (1) can be derived using Theorems 185, 186, and 187, and that the ℓubp comes in in proving Theorem 187. Earlier in the Exploration Exercises you have seen that (2) follows from Theorems 184 and 185.

Having proved (1) and (2), we are justified in introducing the operator '$\sqrt{\ }$' to refer to the inverse of the restricted squaring function, and we do introduce it by adopting the defining principle:

$(*_1)$ $\forall_{x \geq 0} (\sqrt{x} \geq 0 \text{ and } (\sqrt{x})^2 = x)$ [See page 9-5.]

From $(*_1)$ and (2) we have the uniqueness theorem:

$(*_2)$ $\forall_{x \geq 0} \forall_y [(y > 0 \text{ and } y^2 = x) \Rightarrow y = \sqrt{x}]$ [See page 9-5.]

The inverse, $\{(x, y), x \geq 0: y = \sqrt{x}\}$, of the restricted squaring function is called the principal square root function. From Theorems 184 and 185 it follows that the principal square root function is increasing.

Using Theorems 185, 186, and 187 it can be proved that the principal square root function is continuous.

All that we have just said about the principal square root function was based on two very general theorems [Theorems 184 and 187] about inverses and two theorems [Theorems 185 and 186] about positive-integral power functions. The discussion can be repeated with reference to any positive-integral power function restricted to nonnegative arguments. In particular, one can prove:

(1_n) $\quad \forall_n \forall_{y \geq 0} \exists_{x \geq 0}\ x^n = y$ [by Theorems 185, 186, 187]

and:

(2_n) $\quad \forall_n \forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_1^{\,n} = x_2^{\,n} \Rightarrow x_1 = x_2]$ [by Theorems 184, 185]

In geometric terms (1_n) says, for each n, that each horizontal line with nonnegative intercept $[y \geq 0]$ crosses the graph of the restricted nth power function; and (2_n) says that each horizontal line which crosses the graph does so in precisely one point. Since all nth powers of non-negative numbers are nonnegative we may conclude that, for each n, the restricted nth power function has an inverse whose domain is precisely the set of nonnegative numbers. Consequently, we are justified in introducing an operator '$\sqrt[n]{\ }$', to use in referring to these inverse, by adopting the defining principle:

(PR) $\quad \forall_n \forall_{x \geq 0} (\sqrt[n]{x} \geq 0$ and $(\sqrt[n]{x})^n = x)$

From (PR) and (2_n) we have the uniqueness theorem:

$\forall_n \forall_{x \geq 0} \forall_y [(y \geq 0$ and $y^n = x) \Rightarrow y = \sqrt[n]{x}\,]$ [Theorem 188]

And, calling the function $\{(x, y),\ x \geq 0\colon\ y = \sqrt[n]{x}\}$ <u>the principal nth root function</u>, we have, from Theorems 184-187:

Each principal positive-integral root function is continuous and increasing on the set of nonnegative numbers. [Theorem 189]

As in the past, we shall follow the custom of abbreviating '$\sqrt[2]{\ }$' to '$\sqrt{\ }$'. Notice, also, that, for each $x \geq 0$, $\sqrt[1]{x} = x$. [To prove this we use Theorem 188: For $a > 0$, $a > 0$, and $a^1 = a$. So (by Theorem 188), $a = \sqrt[1]{a}$.]

RADICAL EXPRESSIONS

The defining principle (PR) and Theorem 188 give us the necessary tools for manipulating radical expressions.

A radical expression is one which contains a root sign. Here are some examples:

$$2x\sqrt[3]{3x^2}, \qquad \sqrt[5]{\sqrt[3]{11}}, \qquad (\sqrt[4]{2} + \sqrt{2})^4, \qquad \frac{5}{1 - \sqrt[3]{7}}$$

In particular, a radical expression of the form:

$$\sqrt[n]{x}$$

is called a radical. The expression under the root sign is called the radicand of the radical. The number n is called the index of the radical.

Note that the defining principle restricts us to radicands whose values are nonnegative numbers.

Example 1. Show that $\sqrt[3]{54} = \sqrt[3]{27} \cdot \sqrt[3]{2}$.

Solution. By (PR), since $27 \geq 0$ and $2 \geq 0$, $\sqrt[3]{27} \geq 0$ and $\sqrt[3]{2} \geq 0$. So, $\sqrt[3]{27} \cdot \sqrt[3]{2} \geq 0$. Also,

$$(\sqrt[3]{27} \cdot \sqrt[3]{2})^3 = (\sqrt[3]{27})^3 \cdot (\sqrt[3]{2})^3 = 27 \cdot 2 = 54 \geq 0.$$

So, by Theorem 188, $\sqrt[3]{54} = \sqrt[3]{27} \cdot \sqrt[3]{2}$.

Example 2. Show that $\sqrt[6]{5^2} = \sqrt[3]{5}$

Solution. We wish to show that the principal sixth root of 5^2 is $\sqrt[3]{5}$. According to Theorem 188, to do this it is sufficient to show that $\sqrt[3]{5} \geq 0$ and $(\sqrt[3]{5})^6 = 5^2$. By (PR), since $5 \geq 0$, $\sqrt[3]{5} \geq 0$. Also,

$$(\sqrt[3]{5})^6 = (\sqrt[3]{5})^{3 \cdot 2} = [(\sqrt[3]{5})^3]^2.$$

Again by (PR), $(\sqrt[3]{5})^3 = 5$. So, $(\sqrt[3]{5})^6 = 5^2$.

Hence, by Theorem 188, $\sqrt[6]{5^2} = \sqrt[3]{5}$.

Example 3. Show that $\sqrt[3]{\sqrt[4]{7}} = \sqrt[12]{7}$.

Solution. By (PR), since $7 \geq 0$, $\sqrt[4]{7} \geq 0$, and, since $\sqrt[4]{7} \geq 0$, $\sqrt[3]{\sqrt[4]{7}} \geq 0$.

$$\left(\sqrt[3]{\sqrt[4]{7}}\right)^{12} = \left[\left(\sqrt[3]{\sqrt[4]{7}}\right)^3\right]^4 = \left[\sqrt[4]{7}\right]^4 = 7$$

So, by Theorem 188, $\sqrt[3]{\sqrt[4]{7}} = \sqrt[12]{7}$.

Example 4. Show that $(\sqrt[3]{5})^2 = \sqrt[3]{5^2}$.

Solution. Since squares are nonnegative, $(\sqrt[3]{5})^2 \geq 0$.

$$[(\sqrt[3]{5})^2]^3 = [(\sqrt[3]{5})^3]^2 = 5^2$$

So, $(\sqrt[3]{5})^2 = \sqrt[3]{5^2}$.

Example 5. Show that $\sqrt[5]{\frac{3}{16}} = \frac{\sqrt[5]{6}}{2}$.

Solution. $\sqrt[5]{\frac{3}{16}} = \sqrt[5]{\frac{3}{16} \cdot \frac{2}{2}} = \sqrt[5]{\frac{6}{2^5}} = \sqrt[5]{\left(\frac{\sqrt[5]{6}}{2}\right)^5} = \frac{\sqrt[5]{6}}{2}$

EXERCISES

A. Verify.

1. $\sqrt[4]{32} = 2\sqrt[4]{2}$
2. $\sqrt[12]{3^{10}} = \sqrt[6]{3^5}$
3. $\sqrt[5]{\sqrt{3}} = \sqrt[10]{3}$
4. $(\sqrt[5]{7})^4 = \sqrt[5]{7^4}$
5. $(\sqrt[4]{2})^{-3} = \sqrt[4]{2^{-3}}$
6. $\sqrt[3]{\frac{1}{2}} = \frac{\sqrt[3]{4}}{2}$
7. $\sqrt[4]{16} \neq -2$
8. $\forall_m (\sqrt[m]{0} = 0 \text{ and } \sqrt[m]{1} = 1)$ [Theorems 190a, c]

B. True or false?

1. $\forall_{x \geq 0} \sqrt[5]{x^5} = x$
2. $\forall_{x \geq 0} \sqrt[3]{x^6} = x^2$
3. $\forall_x \sqrt[3]{x^6} = x^2$
4. $\forall_x \sqrt[4]{x^{12}} = x^3$
5. $\forall_{x \geq 0} \sqrt[4]{x^{12}} = x^3$
4. $\forall_x \sqrt[4]{x^{12}} = |x^3|$

C. Complete.

1. $\sqrt[7]{9} \cdot \sqrt[7]{12} = \sqrt[?]{108}$
2. $\sqrt[15]{2^3} = \sqrt[?]{2}$
3. $\sqrt[5]{8} \cdot \sqrt[15]{2^3} = \sqrt[5]{?}$
4. $\sqrt[6]{49} = \sqrt[3]{?}$
5. $\sqrt[9]{\sqrt[3]{8}} = \sqrt[?]{8}$
6. $(\sqrt[7]{5})^3 = \sqrt[7]{?}$

D. Prove.

1. $\forall_m \forall_{x>0} \sqrt[m]{x} > 0$ [Theorem 190b]

2. $\forall_m \forall_{x\geq 0} \forall_{y\geq 0} \sqrt[m]{x}\sqrt[m]{y} = \sqrt[m]{xy}$ [Theorem 191a]

3. $\forall_m \forall_n \forall_{x\geq 0} \sqrt[m]{x} = \sqrt[nm]{x^n}$ [Theorem 191b]

4. $\forall_m \forall_n \forall_{x\geq 0} \sqrt[n]{\sqrt[m]{x}} = \sqrt[mn]{x}$ [Theorem 191c]

5. $\forall_j \forall_m \forall_{x>0} (\sqrt[m]{x})^j = \sqrt[m]{x^j}$ [Theorem 191d]

E. Show that each of the following generalizations is FALSE.

1. $\forall_{x\geq 0} \sqrt[4]{\sqrt[3]{x}} = \sqrt[7]{x}$

2. $\forall_{x\geq 0} \sqrt[4]{x}\,\sqrt[3]{x} = \sqrt[7]{x}$

3. $\forall_{x\geq 0} \sqrt[4]{x}\,\sqrt[3]{x} = \sqrt[12]{x}$

4. $\forall_{x\geq 0} \forall_{y\geq 0} \sqrt[4]{x^4 + y^4} = x + y$

F. Expand.

Sample. $(\sqrt[3]{5} + 2)^4$

Solution. $(\sqrt[3]{5} + 2)^4 = (\sqrt[3]{5})^4 + 4(\sqrt[3]{5})^3 \cdot 2^1 + 6(\sqrt[3]{5})^2 \cdot 2^2$
$+ 4(\sqrt[3]{5})^1 \cdot 2^3 + 2^4$
$= 5\sqrt[3]{5} + 40 + 24\sqrt[3]{25} + 32\sqrt[3]{5} + 16$
$= 56 + 37\sqrt[3]{5} + 24\sqrt[3]{25}$

1. $(\sqrt[3]{2} + 1)^3$

2. $\left(\sqrt[3]{a^2} - 1\right)\left(\sqrt[3]{a^2} + 1\right)$

3. $(\sqrt[4]{a} + \sqrt[4]{b})^4$

4. $(\sqrt[4]{7} + 2)^5$

5. $(\sqrt[4]{x} + \sqrt{x})^4$

6. $(\sqrt[6]{2} + x)^3$

☆G. Consider the sequence f defined recursively, for some integer m, by:

$$\begin{cases} f_1 = 1 \\ \forall_n f_{n+1} = (\sqrt[m]{f_n} + 1)^m \end{cases}$$

1. Prove that each term of the sequence is positive.

2. Guess an explicit definition for f and prove your guess by mathematical induction.

3. Compute.

(a) $\sum_{p=1}^{100} f_p$, for m = 1 (b) $\sum_{p=1}^{100} f_p$, for m = 2

H. 1. Find the 10th term of the geometric progression 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$, ...

2. Find the sum of the first ten terms of the geometric progression 1, $\sqrt[5]{3}$, $\sqrt[5]{9}$, ...

3. Insert 4 geometric means between 1 and 2.

4. The 6th term of a geometric progression is 2 and the 16th term is 50.

(a) Find the 26th term. (b) Find the 25th term.

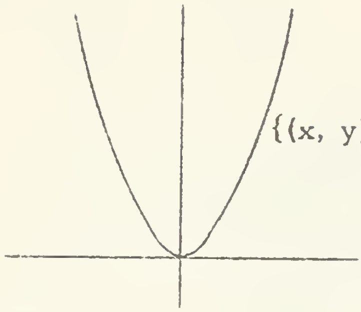
ROOTS OF NEGATIVE NUMBERS

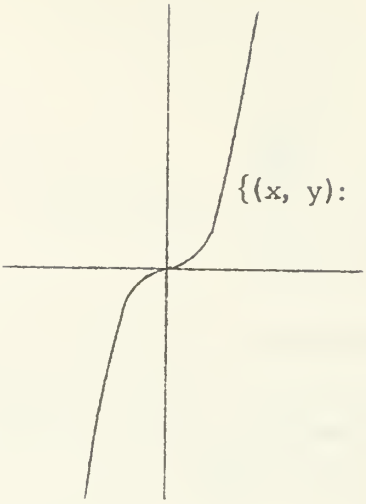
Up to now we have considered roots of nonnegative numbers only [see the defining principle (PR) on page 9-49]. On this basis we know, for example, that the equation '$x^{28} = 5$' has a unique nonnegative solution, namely, $\sqrt[28]{5}$. [And, since $a^2 = b^2$ if and only if ($a = b$ or $a = -b$), it follows that the equation in question has exactly two solutions, $\sqrt[28]{5}$ and $-\sqrt[28]{5}$.]

On the other hand, we know that '$\sqrt[28]{x} = -5$' has no solution [Why?].

As another example, consider the equation '$x^3 = 8$'. Again, our knowledge of the principal root functions assures us that this equation has a unique nonnegative solution. Does it have a negative solution? Does the equation '$x^3 = -8$' have a solution?

The preceding considerations suggest that it may be possible to extend the domain of some of the principal root functions to include negative numbers. Since the domain of a principal root function includes--at most--the members of the range of the associated power function, we can investigate this possibility by reconsidering the positive-integral power functions. As examples, let's look at the squaring function and the cubing function.





As is clear from its graph, the converse of the squaring function is not a function [Explain.] It is for this reason that, in introducing the principal square root function, we defined it as the inverse of the squaring function restricted to nonnegative arguments. However, as is clear from its graph [and as follows by Theorem 185′], the converse of the cubing function is a function--the cubing function, itself, has an inverse. So, we might have defined the principal cube root function to be the inverse of the cubing function--rather than, as we did, as the inverse of $\{(x, y), x \geq 0: y = x^3\}$.

This important difference between the squaring function and the cubing function is expressed in two theorems:

$$\forall_x \ (-x)^2 = x^2 \qquad \text{and:} \qquad \forall_x \ (-x)^3 = -x^3$$

Indeed, using the first it is not difficult to show that, since the squaring function is increasing on $\{x: x \geq 0\}$, it follows that it is decreasing on $\{x: x \leq 0\}$. Using the second we can show that, since the cubing function is increasing on $\{x: x \geq 0\}$, it is also increasing on $\{x: x \leq 0\}$. Consequently [since 0 belongs to both $\{x: x \geq 0\}$ and $\{x: x \leq 0\}$], the cubing function is increasing [everywhere]. It also follows from the second theorem that, since the range of $\{(x, y), x \geq 0: y = x^3\}$ is $\{y: y \geq 0\}$, that of $\{(x, y), x \leq 0: y = x^3\}$ is $\{y: y \leq 0\}$. Consequently, the range of the cubing function is the set of all real numbers.

From these two results it follows that the cubing function has an inverse--say, g--which is an increasing function whose domain is the set of all real numbers. Theorem 187 tells us that g is continuous.

The difference we have noticed between the squaring and cubing functions is merely an instance of a difference between even positive-integral power functions and odd positive-integral power functions. In fact, using the theorem on squares noticed above it is easy to prove:

$$\forall_n \forall_x \ (-x)^{2n} = x^{2n}$$

and, using this, to prove:

$$\forall_n \forall_x \ (-x)^{2n-1} = -x^{2n-1}$$

Using the last, we can repeat, for any odd power function, the remarks made above concerning the cubing function. This being so, it is convenient to adopt an additional defining principle for principal odd roots, supplementary to (PR):

$$\text{(PR}'\text{)} \qquad \forall_n \forall_x \ (\sqrt[2n-1]{x}\,)^{2n-1} = x$$

We then have the theorems:

$$\forall_n \forall_x \forall_y \ [y^{2n-1} = x \Rightarrow y = \sqrt[2n-1]{x}\,] \qquad \text{[Theorem 188}'\text{]}$$

and:

Each principal odd positive-integral root function is continuous and increasing on the set of all real numbers. [Theorem 189′]

Using (PR′) and Theorem 188′ it is easy to prove:

$$\forall_n \forall_x \ \sqrt[2n-1]{-x} = -\sqrt[2n-1]{x} \qquad \text{[Theorem 190}'\text{]}$$

Moreover, the four parts of Theorem 191 now hold for <u>all</u> x in case m and n are both odd.

For example, consider Theorem 191b. For m odd and for any number a,

$$(\sqrt[m]{a}\,)^{nm} = (\sqrt[m]{a}\,)^{mn} = [(\sqrt[m]{a}\,)^m]^n = a^n \qquad \text{[by (PR}'\text{)]}.$$

Now, if n is also odd, it follows that nm is odd and, so, by Theorem 188′, <u>for m and n both odd</u>,

$$\forall_x \ \sqrt[nm]{x^n} = \sqrt[m]{x}\,.$$

As just illustrated in the case of Theorem 191b, the proof of the supplement of each part of Theorem 191 is just like the proof of the part itself

except that (PR′) and Theorem 188′ take the place of (PR) and Theorem 188. [For the resulting theorems, see Theorem 191′ on page 9-359.]

Theorem 191b can be supplemented in another way. Suppose that n is even. Then, for any a, $a^n = |a|^n$. Hence, since $|a| \geq 0$, $\sqrt[nm]{a^n} = \sqrt[nm]{|a|^n} = \sqrt[m]{|a|}$, by Theorem 191b. [The case n = 2, m = 1 should be familiar to you.] So, <u>for n even</u>,

$$\forall_m \forall_x \sqrt[nm]{x^n} = \sqrt[m]{|x|}.$$

Now, it is not hard to prove [using (PR′) and Theorem 188] that, for m odd,

$$\forall_x \sqrt[m]{|x|} = |\sqrt[m]{x}|.$$

Consequently, <u>for n even and m odd</u>,

$$\forall_x \sqrt[nm]{x^n} = |\sqrt[m]{x}|.$$

EXERCISES

<u>A</u>. True or false?

1. $\forall_{x \geq 0} \sqrt[3]{x^3} = x$
2. $\forall_{x \leq 0} \sqrt[3]{x^3} = x$
3. $\forall_x \sqrt[3]{x^3} = x$
4. $\forall_{x \geq 0} \sqrt[5]{-x^5} = x$
5. $\forall_{x \leq 0} \sqrt[5]{-x^5} = x$
6. $\forall_x \sqrt[5]{-x^5} = -x$
7. $\forall_{x \geq 0} \sqrt[4]{x^4} = x$
8. $\forall_{x \leq 0} \sqrt[4]{x^4} = x$
9. $\forall_x \sqrt[4]{x^4} = x$
10. $\forall_x \sqrt[4]{x^4} = |x|$
11. $\forall_{x \leq 0} \sqrt[4]{x^4} = |x|$
12. $\forall_{x \geq 0} \sqrt[4]{x^4} = |x|$
13. $\forall_x \sqrt[5]{x^7} = x\sqrt[5]{x^2}$
14. $\forall_{x \geq 0} \sqrt[6]{x^7} = x \cdot \sqrt[6]{x}$
15. $\forall_x \sqrt[4]{x^{14}} = x^3 \cdot \sqrt[4]{x$
16. $\forall_x \sqrt[6]{x^8} = x\sqrt[6]{x^2}$
17. $\forall_x \sqrt[6]{x^8} = |x| \cdot \sqrt[6]{x^2}$
18. $\forall_x \sqrt[6]{x^8} = x \cdot \sqrt[3]{x}$
19. $\forall_x \forall_y \sqrt[3]{x^5 y^7} = xy^2 \cdot \sqrt[3]{x^2 y}$
20. $\forall_x \forall_y \sqrt[4]{x^{22} y^{14}} = x^5 y^3 \cdot \sqrt[4]{x^2 y^2}$
21. $\forall_x \forall_y \sqrt[6]{x^{14} y^{28}} = x^2 y^4 \cdot \sqrt[6]{x^2 y^4}$
22. $\forall_x \forall_y \sqrt[6]{x^{14} y^{28}} = x^2 y^4 \cdot \sqrt[3]{xy^2}$

* * *

Study these examples of transforming radical expressions.

Example 1. $\sqrt[3]{54a^5b^7} = \sqrt[3]{54} \cdot \sqrt[3]{a^5} \cdot \sqrt[3]{b^7}$

$= \left(3\sqrt[3]{2}\right)\left(a\sqrt[3]{a^2}\right)\left(b^2 \cdot \sqrt[3]{b}\right)$

$= 3ab^2 \cdot \sqrt[3]{2a^2b}$

[Are any restrictions necessary on the values of 'a' and 'b'?]

Example 2. $\sqrt[4]{48a^6b^{10}} = \sqrt[4]{(16a^4b^8)(3a^2b^2)}$

$= 2\,|a|\,b^2 \cdot \sqrt[4]{3a^2b^2}$

[Any restrictions needed?]

Example 3. $\sqrt[4]{48a^5b^{10}} = \sqrt[4]{(16a^4b^8)(3ab^2)}$

$= 2ab^2 \cdot \sqrt[4]{3ab^2}$, $\quad [a \geq 0]$

Example 4. $\sqrt[4]{48a^5b^7} = \sqrt[4]{(16a^4b^4)(3ab^3)}$

$= 2ab \cdot \sqrt[4]{3ab^3}$, $\quad [ab \geq 0]$

Example 5. $\sqrt[3]{\dfrac{-3xy}{8u^3v^3}} = \dfrac{-\sqrt[3]{3xy}}{\sqrt[3]{(2uv)^3}} = \dfrac{-\sqrt[3]{3xy}}{2uv}$, $\quad [uv \neq 0]$

Example 6. $\sqrt[4]{\dfrac{3}{2xy^3}} = \sqrt[4]{\dfrac{3}{2xy^3} \cdot \dfrac{2^3x^3y}{2^3x^3y}}$

$= \sqrt[4]{\dfrac{24x^3y}{(2xy)^4}}$

$= \dfrac{\sqrt[4]{24x^3y}}{2\,|xy|}$, $\quad [xy > 0]$

$= \dfrac{\sqrt[4]{24x^3y}}{2xy}$, $\quad [xy > 0]$

Example 7. $\sqrt[12]{9a^2b^4c^{10}} = \sqrt[12]{(3ab^2c^5)^2}$

$= \sqrt[6]{3ab^2c^5}$, [$ac \geq 0$]

Example 8. $\sqrt[3]{2x^2y} \cdot \sqrt[4]{3xy^3} = \sqrt[12]{(2x^2y)^4} \cdot \sqrt[12]{(3xy^3)^3}$

$= \sqrt[12]{2^4 3^3 x^{11} y^{13}}$

$= |y| \cdot \sqrt[12]{432x^{11}y}$, [$x \geq 0 \leq y$]

$= y \cdot \sqrt[12]{432x^{11}y}$, [$x \geq 0 \leq y$]

* * *

B. 1. Transform by reducing the radicand. [See Examples 1-4.]

(a) $\sqrt[3]{128y^3k^5m^{12}}$ (b) $\sqrt[3]{16xy^3z^7}$ (c) $\sqrt[3]{-16a^9b^8}$

(d) $\sqrt[4]{32a^5b^8c^{17}}$ (e) $\sqrt{36x^{36}}$ (f) $\sqrt[3]{0.001x^{-6}}$

(g) $\sqrt[5]{x^6}$ (h) $\sqrt[3]{x^{11}}$ (i) $\sqrt[3]{-y^7}$

(j) $\sqrt{a^2+2ab+b^2}$ (k) $\sqrt[m]{x^{m+2}y^{m+3}}$ (ℓ) $\sqrt[m]{x^{2m}y^{3m}}$

(m) $\sqrt{x^2(1+3x^2y^2)}$ (n) $3 \cdot \sqrt[3]{x^6 - x^4y^2}$ (o) $\sqrt[3]{64(x-y)^5}$

2. Transform to a radical.

Sample. $3x\sqrt{2y} = \sqrt{(9x^2)(2y)} = \sqrt{18x^2y}$, [$x \geq 0 \leq y$]

(a) $4c^2 \cdot \sqrt{cf}$ (b) $5k \cdot \sqrt[3]{k}$ (c) $x \cdot \sqrt[n]{x}$ (d) $x^2y^3 \cdot \sqrt[n]{xy}$

(e) $\frac{1}{3t}\sqrt{10t}$ (f) $3a^2|b| \cdot \sqrt[4]{5a^2b^2}$ (g) $0.4 \cdot \sqrt[5]{100}$ (h) $-2 \cdot \sqrt[3]{5}$

3. Transform by reducing the index. [See Examples 5 and 7.]

(a) $\sqrt[4]{49y^2}$ (b) $\sqrt[6]{9x^4}$ (c) $\sqrt[6]{27t^3}$ (d) $\sqrt[12]{a^2b^4c^6}$

(e) $\sqrt[4]{\frac{49a^2}{225x^2}}$ (f) $\sqrt[4]{a^2+2ab+b^2}$ (g) $\sqrt[7n]{x^{4n}y^{2n}}$

4. Transform to an expression which contains a single radical. [See Example 8.]

(a) $\sqrt[3]{5} \cdot \sqrt[4]{3}$ (b) $\sqrt[3]{y} \cdot \sqrt[4]{y}$ (c) $7\sqrt{x} \cdot 6\sqrt[3]{3}$

(d) $\sqrt[3]{3a^2b} \cdot \sqrt{5a^3b^2}$ (e) $\sqrt[4]{2x^2y^3} \cdot \sqrt[5]{2x^4y^2}$

(f) $\sqrt[4]{2x^2y^3z} \div \sqrt[3]{3xyz^4}$ (g) $12x^2y \cdot \sqrt[3]{12x^2y} \div (3xy^2 \cdot \sqrt{2xy^3})$

5. Transform to an expression in which the radicand does not contain a fraction. [See Example 6.]

(a) $\sqrt[3]{\frac{1}{a}}$ (b) $\sqrt{\frac{2x}{3y}}$ (c) $\sqrt[5]{\frac{1}{4y^4}}$

(d) $\sqrt[3]{\frac{8t}{9s^2}}$ (e) $\sqrt{\frac{a}{b^2} + \frac{b}{a^2}}$ (f) $\sqrt[3]{\frac{1}{a^2} + \frac{1}{b^2}}$

6. If $f(x) = x^3 - 8x^2 + 19x - 14$, show that $f(3 - \sqrt{2}) = 0$.

☆C. Study the following true equations:

$$\sqrt{2\tfrac{2}{3}} = \sqrt{\tfrac{8}{3}} = \sqrt{4 \cdot \tfrac{2}{3}} = 2 \cdot \sqrt{\tfrac{2}{3}}$$

$$\sqrt{3\tfrac{3}{8}} = \sqrt{\tfrac{27}{8}} = \sqrt{9 \cdot \tfrac{3}{8}} = 3 \cdot \sqrt{\tfrac{3}{8}}$$

$$\sqrt{4\tfrac{4}{15}} = 4 \cdot \sqrt{\tfrac{4}{15}}$$

$$\sqrt{5\tfrac{5}{24}} = 5 \cdot \sqrt{\tfrac{5}{24}}$$

1. Generalize.

2. Does the generalization you made in Exercise 1 include cases such as the following? If not, generalize further.

$\sqrt[3]{2\tfrac{2}{7}} = 2 \cdot \sqrt[3]{\tfrac{2}{7}}$; $\sqrt[3]{3\tfrac{3}{26}} = 3 \cdot \sqrt[3]{\tfrac{3}{26}}$;

$\sqrt[3]{4\tfrac{4}{63}} = 4 \cdot \sqrt[3]{\tfrac{4}{63}}$; $\sqrt[4]{4\tfrac{4}{255}} = 4 \cdot \sqrt[4]{\tfrac{4}{225}}$;

$\sqrt[5]{2\tfrac{2}{31}} = 2 \cdot \sqrt[5]{\tfrac{2}{31}}$; $\sqrt[10]{3\tfrac{3}{59048}} = 3 \cdot \sqrt[10]{\tfrac{3}{59048}}$

MISCELLANEOUS EXERCISES

1. Of the 75 boys in a small high school, 39 were on the football squad, 14 were on the basketball squad, and 27 did not take part in athletics at all. How many were on both the basketball and football squads?

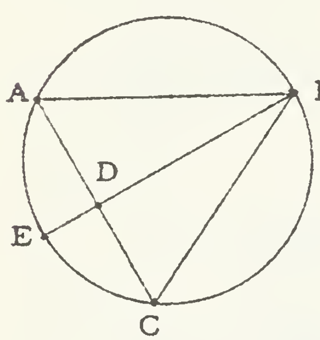
2. Solve the equation: $\sqrt{0.49x^2} = 7$

3. Simplify.

   (a) $\frac{7}{ab + b^2} + \frac{8}{a^2 + ab}$ (b) $\frac{t}{s + t} \div \frac{s}{s^2 - t^2}$

4. If a quart of water is approximately 60 cubic inches in volume and a cubic foot of water weighs approximately 60 pounds, about how many pounds does a gallon of water weigh?

5.



<u>Given:</u> $\overset{\frown}{AB} \cong \overset{\frown}{BC}$,

$\overset{\frown}{AEC}$ is an arc of $120°$,

$\overleftrightarrow{EB}$ is a diameter

<u>Find:</u> the ratio of AB to AD

6. Solve the equation: $\frac{3}{4}x - \frac{1}{2}x - 1 = 19$

7. Simplify.

   (a) $\dfrac{2a - \frac{b + 2}{6}}{\frac{b + 2}{3} - 3a}$ (b) $\dfrac{\frac{1}{x} \cdot \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}}$ (c) $\dfrac{\frac{9}{25} - \left(\frac{3}{5}\right)^2}{\frac{1}{2}}$

8. A certain type of alloy is made of 7 parts copper to 3 parts zinc. How many pounds of zinc will be needed to make 830 pounds of this alloy?

9. Derive a formula for the area-measure K of a circle in terms of its circumference C.

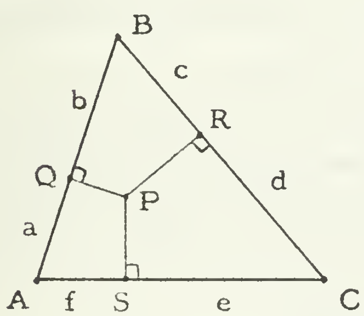
10. Suppose that the sum of two integers is 45. If the ratio of the larger to the smaller is greater than 2 and the ratio of the smaller to the larger is greater than 7/16, what are the integers?

11. Find the dimensions of the rectangle with the largest area-measure that can be cut from a trapezoidal region whose bases are 3.5 and 1 and whose altitude is 10.

12. Solve the system: $\begin{cases} 10^{m+n} = 10000 \\ 10^{m-n} = 100 \end{cases}$

13. If the measures of three angles of a convex pentagon are 100, 118, and 96, respectively, and if the other two angles are congruent, what is the measure of one of them?

14. Suppose that x and y are two-digit numbers. If the digits of one of them are the digits of the other and if their ratio is 5 to 6, what are the numbers?

15.


Hypothesis: $P \in \triangle ABC$,
$\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{PR} \perp \overleftrightarrow{BC}$,
$\overleftrightarrow{PS} \perp \overleftrightarrow{AC}$

Conclusion: $a^2 + c^2 + e^2 = b^2 + d^2 + f^2$

16. Find the first term and common difference of the arithmetic progression whose 27th term is 186 and whose 44th term is 305.

17. The sum of the first four terms of an arithmetic progression is 44 and the fourth term is 17. What is the second term?

18. Expand and express without referring to negative exponents.

(a) $(a^2b^{-2})^3$ (b) $(-a^3b^{-1}c^{-2})^5$ (c) $(a^{-1}b^2c^3)^{-6}$

9.05 <u>The rational numbers</u>.--In Unit 4 you learned a classification of real numbers:

$$\text{real numbers}\begin{cases}\text{rationals [R]}\begin{cases}\text{integers [I]}\begin{cases}\text{positive integers } [I^+]\\ 0\\ \text{negative integers}\end{cases}\\ \text{nonintegral rationals}\end{cases}\\ \text{irrationals}\end{cases}$$

For example:

> 1, 2, 3, etc. are positive integers and -1, -2, -3 are negative integers; the positive integers, 0, and the negative integers make up the set of integers; the rational numbers are those which are quotients of integers by positive integers--
>
> $\frac{4}{5}$, $-\frac{369}{12}$, $\frac{10}{2}$, $\sqrt{2}\cdot\sqrt{8}$, 6, 0.3, $0.\bar{9}$, and $0.8\bar{3}$
>
> are rational numbers; the remaining real numbers are called <u>irrational</u> numbers--$\sqrt{8}$, $\sqrt[3]{9}$, and π are irrational numbers.

In Unit 7 we adopted three basic principles, (I^+_1) - (I^+_3) to characterize the set I^+ of positive integers, and one basic principle (I) to characterize the set I of integers. We now need a basic principle to allow us to prove theorems about the set R of rational numbers.

What we need is a principle which says which real numbers are rational. One common definition is the one mentioned above--that a rational number is a real number which is the quotient of some integer by some positive integer. More briefly,

$$\forall_x\, [x \in R \Longleftrightarrow \exists_k \exists_n\ x = \frac{k}{n}].$$

By the principle of quotients, if $x = \frac{k}{n}$ then $xn = k$. And, by the division theorem [Theorem 49], if $xn = k$ then $x = \frac{k}{n}$. So, the foregoing possible definition is equivalent to:

(R) $$\forall_x\, [x \in R \Longleftrightarrow \exists_n\ xn \in I]$$

In our dealings with rational numbers, we shall use (R) as a basic principle.

Now, how does (R) work? Let's suppose that someone has picked a real number and is wondering if it is rational. A mechanical procedure

for testing it would be to multiply it by 1, then by 2, then by 3, etc. If one [or more] of the products is an integer then the number is rational. If the number is irrational, it will be impossible to get an integral product.

Of course, such a mechanical procedure is never necessary. For example, to show that the real number $\frac{22}{7}$ is rational, think of the principle of quotients. Since $\frac{22}{7} \cdot 7 = 22$ and since $7 \in I^+$ and $22 \in I$, (R) tells us that $\frac{22}{7}$ is rational. Instead of 7, we can use 14 to test the rationality of $\frac{22}{7}$. What other numbers can be used?

Similarly, -5 is a rational number because $-5 \cdot 1 = -5$ and $1 \in I^+$ and $-5 \in I$. What other positive integral multipliers can be used to test the rationality of -5?

Sometimes, the problem can be more complicated. For example, consider $\frac{\sqrt{8}}{3\sqrt{2}}$. Is this number rational or not? The principle of quotients may suggest multiplying by $3\sqrt{2}$, but since $3\sqrt{2} \notin I^+$, doing so would be irrelevant. However,

$$\frac{\sqrt{8}}{3\sqrt{2}} = \frac{2}{3} \qquad \text{[Why?]}$$

and $\frac{2}{3}$ is rational [Explain.].

We can also use (R) in proving closure theorems for the set of rational numbers. For example, let's prove that the set R is closed with respect to addition, that is, that each sum of rational numbers is rational. Suppose that r and s are rational numbers. It follows from (R) that there are positive integers--say, p and q--such that $rp \in I$ and $sq \in I$. Since I is closed with respect to multiplication, it follows that $rpq \in I$ and $sqp \in I$. Since I is closed with respect to addition,

$$rpq + sqp \in I,$$

that is,

$$(r + s)(pq) \in I.$$

Since I^+ is closed with respect to multiplication, $pq \in I^+$. So, by (R), $r + s$ is a rational number.

So far, we have used (R) to prove that numbers are rational. Let's use it, now, to prove that $\sqrt{2}$ is <u>not</u> rational. To do this, let's suppose that $\sqrt{2}$ is rational--that is [by (R)], suppose that there is a positive

integer whose product with $\sqrt{2}$ is integral. Then, by the least number theorem [Theorem 108], there must be a smallest such positive integer --say q. Then, $\sqrt{2}\cdot q$ is integral and, if $p < q$ then $\sqrt{2}\cdot p$ is not integral.

Now, since $1^2 < (\sqrt{2})^2 < 2^2$, it follows [since the principal square root function is increasing] that

$$1 < \sqrt{2} < 2.$$

Since $q > 0$,

$$1q < \sqrt{2}\cdot q < 2q,$$

and, so,

$$(*) \qquad 0 < \sqrt{2}\cdot q - q < q.$$

Now, since $\sqrt{2}\cdot q$ and q are both integers and I is closed with respect to subtraction, $\sqrt{2}\cdot q - q$ is an integer. So, from (*), $\sqrt{2}\cdot q - q$ is a positive integer less than q. Hence, by the choice of q,

$$(1) \qquad \sqrt{2}\,(\sqrt{2}\cdot q - q) \notin I.$$

But,

$$\sqrt{2}\,(\sqrt{2}\cdot q - q) = 2q - \sqrt{2}\cdot q.$$

Since 2 and q are integers and I is closed with respect to multiplication, $2q \in I$. By hypothesis, $\sqrt{2}\cdot q \in I$. Since I is closed with respect to subtraction, $2q - \sqrt{2}\cdot q \in I$. Hence,

$$(2) \qquad \sqrt{2}\,(\sqrt{2}\cdot q - q) \in I.$$

So, the assumption that there is a positive integer whose product with $\sqrt{2}$ is integral has led us to the contradictory conclusions (1) and (2). Consequently, it is not the case that there is a positive integer whose product with $\sqrt{2}$ is integral. That is, by (R), $\sqrt{2}$ is not rational.

EXERCISES

A. 1. Prove that $\frac{48}{97}$, $-\frac{17}{2}$, $\frac{1}{2}\cdot\frac{50}{100}$, 0.496, $\frac{2}{-5}$, $\frac{0}{\pi}$, $\frac{\sqrt{27}}{2\sqrt{3}}$, and $\frac{3\pi}{4\pi}$ are rational.

2. Prove that $\frac{48}{97} + -\frac{17}{2}$ is rational.

3. Prove that each integer is rational.

4. (a) Mildred picks a real number, multiplies it by an integer, and gets an integral product. Does it follow that the number she picked is rational?

   (b) Sarah picks a real number, multiplies it by a positive integer, and gets a nonintegral product. Does it follow that the number she picked is irrational?

5. Prove [we use 'r', 's', and 't' as variables whose domain is R].

   (a) $\forall_r\ -r \in R$ (b) $\forall_r \forall_s\ rs \in R$ (c) $\forall_r \forall_{s \neq 0}\ \frac{r}{s} \in R$

   [Theorems 192a, d, and e.]

6. True or false?

   (a) $\forall_r \forall_x\ [r + x \in R \Rightarrow x \in R]$ (b) $\forall_r \forall_s\ r^2 + s^2 \in R$

   (c) $\forall_r \forall_s\ r^2 + 2rs + s^2 \in R$ (d) $\forall_r \forall_x\ [rx \in R \Rightarrow x \in R]$

   (e) $\forall_{r \geq 0}\ \sqrt{r} \in R$ (f) $\forall_n\ \sqrt{n} \in R$ (g) $\forall_n \exists_{m > 1}\ \sqrt[n]{m} \in R$

B. 1. Prove that $\sqrt{3}$ is irrational.

2. Guess which of these are rational.

   (a) $\sqrt[3]{2}$ (b) $\sqrt[3]{8}$ (c) $\sqrt[7]{1}$ (d) $\sqrt[5]{32}$ (e) $\sqrt{65}$

*

You have probably already guessed that, for each m and n, $\sqrt[n]{m}$ is either an integer or an irrational number. That is, $\sqrt[n]{m}$ is rational only if m is a perfect nth power [m is a perfect nth power if and only if it is the nth power of some integer]. In Appendix B [pages 9-231 through 9-249] your guess is proved correct:

> **Theorem 193.**
>
> $\forall_n \forall_m\ \sqrt[n]{m}$ is irrational unless m is a perfect nth power.

*

*

3. Which of these are rational?

(a) $\sqrt[7]{2^{14}}$ (b) $\sqrt[7]{2^{15}}$ (c) $\sqrt[3]{\frac{1}{8}}$ (d) $\sqrt[5]{.00001}$ (e) $\sqrt[3]{-64}$

<u>Sample.</u> Is $\sqrt{2} + \sqrt{3}$ rational?

<u>Solution.</u> Suppose that $\sqrt{2} + \sqrt{3} \in R$. Then, by (R), there is a positive integer, say q, such that

$$(\sqrt{2} + \sqrt{3})q \in I.$$

Since I is closed with respect to squaring,

$$[(\sqrt{2} + \sqrt{3})q]^2 \in I.$$

That is,

$$(5 + 2\sqrt{6})q^2 \in I.$$

Since $q^2 \in I^+$, it follows that $5 + 2\sqrt{6}$ is rational. So, since $5 \in R$ and $2 \in R$, so does $\sqrt{6}$ [Explain.]. But, by Theorem 193, since 6 is not a perfect square, $\sqrt{6} \notin R$. Hence, $\sqrt{2} \notin R$.

4. Which of these are rational?

(a) $7 - \sqrt{2}$ (b) $4\sqrt{2} - \sqrt{32}$ (c) $\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3}$ (d) $3\sqrt{12} + 2\sqrt{3}$

☆5. (a) Find rational numbers r and s such that $r\sqrt{2} + s\sqrt{3} \in R$. How many pairs can you find?

(b) [For any real numbers a and b, <u>the rational-linear combiations</u> of a and b are the numbers ra + sb, where r and s are rational numbers.] You have seen that the only rational-linear combination of $\sqrt{2}$ and $\sqrt{3}$ which is a rational number is 0. Investigate the conditions on m and n and on r and s under which $r\sqrt{m} + s\sqrt{n}$ is irrational.

<u>C</u>. 1. Prove that the real numbers are <u>dense</u>. That is, prove that beween each two real numbers there is another real number:

$$\forall_x \forall_{y > x} \exists_z \; x < z < y$$

[<u>Hint</u>. How about the average of the given numbers?]

2. Prove that the rational numbers are dense. That is, prove that

$$\forall_r \forall_{s > r} \exists_t \; r < t < s.$$

3. What theorem tells you that the positive integers are not dense?

* * *

It is perhaps not very surprising that the rational numbers are dense. But, since [as shown in Appendix B, see Theorem 200] there are many more irrational numbers than there are rational numbers, most people do find it surprising that the rational numbers are dense <u>in</u> the real numbers--that is, that between any two real numbers there is a rational number:

> Theorem 201.
>
> $$\forall_x \forall_{y > x} \exists_r \; x < r < y$$

For a proof of this, see Appendix B.

* * *

4. Use Theorem 201 to prove:

$$\forall_x \forall_{y > x} \exists_r \exists_s \; x < r < s < y$$

☆D. The reciprocating operation is defined by:

$$/x = 1 \div x, \quad \text{for } x \neq 0$$

Prove the following theorems.

1. $\forall_{x \neq 0} \; x \cdot /x = 1$

2. $\forall_x \forall_{y \neq 0} \; x \cdot /y = x \div y$

3. $\forall_{x \neq 0} \forall_{y \neq 0} \; /x \cdot /y = /(xy)$

4. $\forall_{x \neq 0} \; //x = x$

5. Reciprocating is not distributive with respect to addition.

6. $\forall_x \forall_y \forall_{z \neq 0} \; (x + y) \cdot /z = x \cdot /z + y \cdot /z$

7. R is closed with respect to reciprocation.

8. $\forall_{x \neq 0} \forall_{y \neq 0} \; /x \div /y = /(x \div y)$

9. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; (xz) \cdot /(yz) = x \cdot /y$

9.06 Rational exponents. --In Unit 8 you learned about the use of integers as exponents. We began with nonnegative integral exponents:

(1) $$\forall_x \forall_{k \geq 0}\ x^k = \prod_{p=1}^{k} x$$

and then introduced negative integral exponents:

(2) $$\forall_{x \neq 0} \forall_{k < 0}\ x^k = \frac{1}{x^{-k}}$$

So, for example, (1) and the recursive definition of Π-notation tell us that $2^3 = 2^2 \cdot 2 = 2 \cdot 2 \cdot 2$, and that, for each x, $x^0 = 1$. Since $2^3 = 8$ [and since $2 \neq 0$ and $-3 < 0$] (2) tells us that $2^{-3} = 1/2^{--3} = 1/2^3 = 1/8$.

Using these two definitions we were able to prove a number of theorems which justified simple tricks for manipulating exponential expressions. For example,

by Theorem 155, $(\sqrt{3})^{-5}(\sqrt{3})^3 = (\sqrt{3})^{-5+3} = (\sqrt{3})^{-2}$,

by Theorem 154, $(\sqrt{3})^{-2} = 1/(\sqrt{3})^2 = 1/3$ [by (PR)],

by Theorem 156, $\pi^6/\pi^{-4} = \pi^{6--4} = \pi^{10}$,

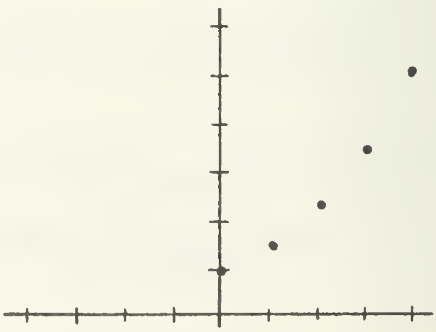
by Theorem 157, $[(-1.5)^2]^3 = (-1.5)^6$,

by Theorem 158, $(\pi\sqrt{2})^4 = \pi^4(\sqrt{2})^4$ [$= 4\pi^4$],

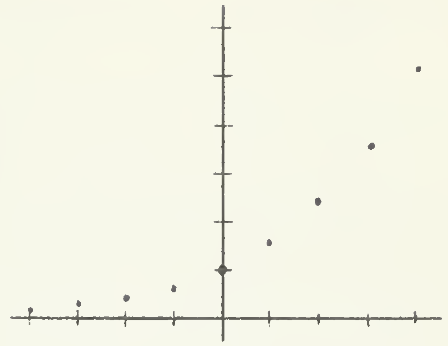
by Theorem 160, $(2/3)^{-3} = (3/2)^3$, and,

by Theorem 159, $(3/2)^3 = 3^3/2^3$.

In Unit 8, we spoke of (1) as defining, for each x, the exponential sequence with base x. Each such sequence is a function whose domain is the set of nonnegative integers.



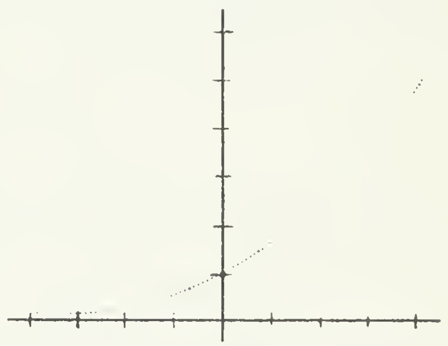
Definition (2) shows how to extend each exponential sequence with nonzero base to a function whose domain is the set of all integers. These "exponential functions with integral arguments" have the desirable properties, some of which are exemplified above, which are described in Theorems 154-160. Those with positive bases have the additional properties described in Theorems 152a and 161.



In this section we shall discover meanings for expressions like:

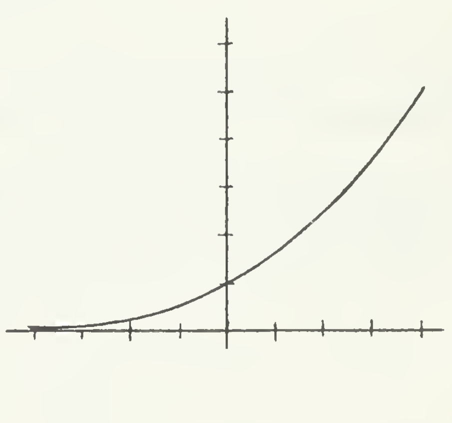
$$3^{\frac{1}{2}} \qquad 6^{\frac{3}{5}} \qquad 2^{24/10} \qquad \pi^{-0.8\overline{3}}$$

which will enable us to extend, further, the exponential sequences with positive bases to "exponential functions with rational arguments" in such a way that these functions will also have the properties described in Theorem 152a and Theorems 154-161. For example, it will be the case that



$$2^{1/2} \cdot 2^{3/5} = 2^{11/10} \quad \text{and that} \quad (\pi^{-0.8\overline{3}})^3 = \pi^{-2.5}.$$

Later, we shall see that each of these new functions is monotonic and continuous on its domain R, and we shall see how these properties can be used to define an extension of each such function to an exponential function whose domain is the set of all real numbers. Theorems analogous to those previously mentioned will again hold, and these final exponential functions will turn out, also, to be monotonic and continuous--this time, on the set of all real numbers.



LOOKING FOR A DEFINITION

Our problem is to assign meanings to expressions like:

$$3^{\frac{1}{2}} \qquad 6^{\frac{3}{5}} \qquad 2^{24/10} \qquad \pi^{-0.8\bar{3}}$$

in such a way that the laws of exponents which we have proved for integral exponents will continue to hold. We faced a similar problem in Unit 8 when we sought to assign meanings to expressions like '2^{-3}'. We solved <u>that</u> problem by, first, discovering what meanings would have to be assigned to such expressions <u>if</u> one of the laws [the addition law] were to hold and, second, showing that if meanings were assigned in this way then this law and the others <u>would</u> hold. A similar procedure will be effective here.

If the multiplication law for exponents is to hold when exponents are rational numbers, as well as when only integers are allowed as exponents, it must be the case that

$$(3^{1/2})^2 = 3^{\frac{1}{2}\cdot 2} = 3^1 = 3$$

--that is, it must be the case that $3^{\frac{1}{2}}$ is a number whose square is 3. As we know, there are just two such numbers, $\sqrt{3}$ and $-\sqrt{3}$. We could ensure that $\left(3^{1/2}\right)^2 = 3^{\frac{1}{2}\cdot 2}$ by agreeing either that $3^{\frac{1}{2}} = \sqrt{3}$ or that $3^{\frac{1}{2}} = -\sqrt{3}$. Which agreement should we make? One way to decide is to consider '$3^{\frac{1}{4}}$'. We must be able to assign such a meaning to this expression that

$$(3^{1/4})^2 = 3^{\frac{1}{4}\cdot 2} = 3^{\frac{1}{2}}.$$

To be able to do so, since squares of real numbers are nonnegative, $3^{\frac{1}{2}}$ must be nonnegative. So, the only agreement as to '$3^{\frac{1}{2}}$' which has a chance of being satisfactory is that

$$3^{\frac{1}{2}} = \sqrt{3}.$$

Now, let's consider '$6^{\frac{3}{5}}$'. If the multiplication law is to hold,

$$(6^{3/5})^5 = 6^{\frac{3}{5}\cdot 5} = 6^3$$

--that is, $6^{\frac{3}{5}}$ must be a number whose 5th power is 6^3. There is only one such number, $\sqrt[5]{6^3}$. [Explain why $-\sqrt[5]{6^3}$ is not such a number.] So, our only hope is to agree that

$$6^{\frac{3}{5}} = \sqrt[5]{6^3}.$$

Finally, let's consider '$2^{24/10}$'. Since $\frac{24}{10}\cdot 10 = 24$, the same kind of argument we have used before shows us that our only hope is to decide that

$$2^{24/10} = \sqrt[10]{2^{24}}.$$

This last example may suggest a possible source of difficulty. Not only is $\frac{24}{10}\cdot 10 = 24$, but, also, $\frac{24}{10}\cdot 15 = 36$. So, the same line of reasoning prescribes that we agree that

$$2^{24/10} = \sqrt[15]{2^{36}}.$$

Clearly, our program will run into difficulties unless

$$\sqrt[15]{2^{36}} = \sqrt[10]{2^{24}}.$$

Let's check:

$$\sqrt[15]{2^{36}} = \sqrt[3\cdot 5]{(2^{12})^3} = \sqrt[5]{2^{12}} = \sqrt[2\cdot 5]{(2^{12})^2} = \sqrt[10]{2^{24}} \qquad \text{[Theorem 191b.]}$$

Apparently things are going to work out all right. If r is a rational number and m is a positive integer for which $rm \in I$, we can, for any $a > 0$, agree that

$$a^r = \sqrt[m]{a^{rm}}.$$

This makes sense because, for any $a \neq 0$ and any integer rm, a^{rm} is, by (1) and (2), a real number which, if $a > 0$, is, by Theorem 152a, a positive number. And, by the defining principle (PR), $\sqrt[m]{a^{rm}}$ is, in this case, a real number. Moreover, if n is any other positive integer for which $rn \in I$ then

$$\sqrt[n]{a^{rn}} = \sqrt[mn]{a^{rnm}} = \sqrt[nm]{a^{rmn}} = \sqrt[m]{a^{rm}}.$$

Theorem 191b

So, however we use this agreement, we shall always be led to assigning the same value to any numeral of the form 'a^r'--the agreement is, we may say, self-consistent.

Before adopting this agreement as a definition, we have still to check another consistency question. To see what this question is, notice that, since 2, say, is both a rational number and a positive integer, it follows by the definition (1) for exponential sequences, that $3^2 = 3 \cdot 3$. Since $2 \cdot 1 \in I$ [so that 2 is a rational number], it follows from the proposed agreement that $3^2 = \sqrt[1]{3^2}$, where the meaning of the second '3^2' is to be determined, as above, from (1). So, according to the proposed agreement, $3^2 = \sqrt[1]{3 \cdot 3}$. Since, for each x, $\sqrt[1]{x} = x$, it follows that $\sqrt[1]{3 \cdot 3} = 3 \cdot 3$. So, both (1) and the proposed agreement assign the same value to '3^2'. Generalizing on this argument shows that the proposed agreement is consistent with both the earlier definitions (1) and (2).

The upshot of these checks for consistency is that no harm will come from adopting the proposed agreement as a supplement to (1) and (2). We shall do so, after using Theorem 191d to transform it into a slightly more convenient form:

$$(3) \qquad \forall_{x>0} \forall_r \forall_{m,\ \underset{\uparrow\ \text{[read 'with']}}{}\ rm \in I}\ x^r = (\sqrt[m]{x})^{rm}$$

[The restriction on 'm' is needed to ensure that (1) and (2) make sense of '$(\sqrt[m]{x})^{rm}$'.]

Up to now we have rather neglected the exponential sequence with base 0. In Unit 8 we saw that it was futile to attempt to define negative-integral powers of 0, and the same argument applies to discourage us from defining other negative-rational powers of 0. However, it seems [and will turn out to be] reasonable to adopt:

$$(3') \qquad 0^0 = 1 \qquad \text{and} \qquad \forall_{r>0}\ 0^r = 0$$

EXERCISES

A. Definition (3) enables us to translate an expression for a rational power into an expression for an integral power.

Examples: $3^{\frac{2}{5}} = (\sqrt[5]{3})^2 = (\sqrt[10]{3})^4 = \ldots$

$$5^{\frac{1}{7}} = (\sqrt[7]{5})^1 = \sqrt[7]{5} = (\sqrt[14]{5})^2 = \ldots$$

Complete each of the following.

1. $2^{\frac{2}{3}} = (\sqrt[3]{2})^? = (\sqrt[12]{2})^? = (\sqrt[?]{2})^6 = (\sqrt[?]{4})^6$

2. $8^{\frac{1}{5}} = (\sqrt[5]{8})^? = (\sqrt[10]{8})^? = (\sqrt[20]{8})^? = (\sqrt[?]{2})^?$

3. $(\sqrt[5]{7})^2 = 7^?$

4. $\sqrt[6]{9^3} = 9^? = 3^?$

B. Later we shall prove that definition (3) enables us to justify all of the familiar laws of exponents which we have already established for integral exponents. In this exercise, we shall justify the laws as they apply to special cases.

Sample 1. Show that $3^{-\frac{2}{3}} = \dfrac{1}{3^{2/3}}$

Solution.

$$\begin{aligned} 3^{-\frac{2}{3}} &= 3^{\frac{-2}{3}} \\ &= \left(\sqrt[3]{3}\right)^{-2} \\ &= \frac{1}{\left(\sqrt[3]{3}\right)^2} \\ &= \frac{1}{3^{2/3}} \end{aligned}$$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ Theorem 154

Sample 2. Show that $4^{\frac{2}{3}} \cdot 4^{\frac{3}{5}} = 4^{\frac{19}{15}}$.

Solution.

$$\begin{aligned} 4^{\frac{2}{3}} \cdot 4^{\frac{3}{5}} &= 4^{\frac{10}{15}} \cdot 4^{\frac{9}{15}} \\ &= \left(\sqrt[15]{4}\right)^{10} \cdot \left(\sqrt[15]{4}\right)^9 \\ &= \left(\sqrt[15]{4}\right)^{19} \\ &= 4^{\frac{19}{15}} \end{aligned}$$

Sample 3. Show that $\left(7^{-\frac{2}{5}}\right)^{\frac{3}{7}} = 7^{-\frac{6}{35}}$

Solution.
$$\left(7^{-\frac{2}{5}}\right)^{\frac{3}{7}} = \left[\left(\sqrt[5]{7}\right)^{-2}\right]^{\frac{3}{7}}$$
$$= \left[\sqrt[7]{\left(\sqrt[5]{7}\right)^{-2}}\right]^{3}$$
$$= \left[\left(\sqrt[7]{\sqrt[5]{7}}\right)^{-2}\right]^{3} \quad \text{Theorem 191d}$$
$$= \left[\sqrt[7]{\sqrt[5]{7}}\right]^{-6} \quad \text{Theorem 157}$$
$$= \left(\sqrt[35]{7}\right)^{-6} = 7^{-\frac{6}{35}}$$

Sample 4. Show that $(2 \cdot 5)^{\frac{2}{3}} = 2^{\frac{2}{3}} \cdot 5^{\frac{2}{3}}$.

Solution.
$$(2 \cdot 5)^{\frac{2}{3}} = \left(\sqrt[3]{2 \cdot 5}\right)^2$$
$$= \left(\sqrt[3]{2} \cdot \sqrt[3]{5}\right)^2 \quad \text{Theorem 191a}$$
$$= \left(\sqrt[3]{2}\right)^2 \cdot \left(\sqrt[3]{5}\right)^2 \quad \text{Theorem 158}$$
$$= 2^{\frac{2}{3}} \cdot 5^{\frac{2}{3}}$$

Establish each of the following.

1. $3^{-\frac{2}{5}} = \dfrac{1}{3^{2/5}}$

2. $4^{\frac{1}{3}} = \dfrac{1}{4^{-1/3}}$

3. $32^{-0.8} = 0.0625$

4. $3^{\frac{4}{7}} \cdot 3^{\frac{2}{3}} = 3^{\frac{26}{21}}$

5. $64^{\frac{2}{3}} \cdot 64^{\frac{1}{2}} = 64^{\frac{7}{6}} = 128$

6. $11^{-\frac{1}{5}} \cdot 11^{\frac{4}{5}} \cdot 11^{\frac{2}{5}} = 11$

7. $7^{\frac{2}{9}} \div 7^{\frac{3}{8}} = 7^{-\frac{11}{72}}$

8. $\left(3^{\frac{2}{5}}\right)^{\frac{4}{7}} = 3^{\frac{8}{35}}$

9. $\left(3^{\frac{4}{5}}\right)^{\frac{2}{7}} = 3^{\frac{8}{35}}$

10. $\left(5^{0.1}\right)^{0.1} = 5^{0.01}$

11. $(6 \cdot 7)^{\frac{3}{4}} = 6^{\frac{3}{4}} \cdot 7^{\frac{3}{4}}$

12. $(16 \cdot 27)^{\frac{2}{3}} = 36 \cdot \sqrt[3]{4}$

13. $\left(2^{\frac{3}{2}} \cdot 6^{\frac{2}{5}}\right)^{10} = 2^{19} \cdot 3^4$

14. $\left(7^2 \cdot 11\right)^{\frac{1}{6}} = \sqrt[3]{7}\ \sqrt[6]{11}$

15. $\left(2^7 \div 3^5\right)^{\frac{1}{35}} = \sqrt[5]{2} \div \sqrt[7]{3}$

16. $\left(\frac{2}{3}\right)^{-\frac{2}{3}} = \left(\frac{3}{2}\right)^{\frac{2}{3}}$ 17. $5^{\frac{2}{3}} > 0$ 18. $5^{-\frac{2}{3}} > 0$

TESTING OUR DEFINITION

We have seen that adopting:

$$(3) \qquad \forall_{x>0} \forall_r \forall_{m,\ rm \in I}\ x^r = (\sqrt[m]{x})^{rm}$$

will not get us into any trouble, and that (3) is the only definition from which we can hope to get satisfactory theorems. The preceding exercises are good evidence that it is possible to prove analogues of Theorems 152a and 154-161. We shall now state these analogous theorems and prove them by the methods you probably used in solving the exercises. [The theorem numbers listed are those which will be given to the final generalizations, to be proved later, in which 'r' and 's' are replaced by variables whose domain is the set of all real numbers. The bracketed versions are analogous to the unbracketed ones, but cover the case of base 0. In this case, as in the definition:

$$(3') \qquad 0^0 = 1 \text{ and } \forall_{r>0}\ 0^r = 0,$$

the exponents must be nonnegative.]

Theorem 203. $\forall_{x>0} \forall_r\ x^r > 0$ $[\forall_{x\geq 0} \forall_{r\geq 0}\ x^r \geq 0]$

Theorem 204. $\forall_{x>0} \forall_r\ x^{-r} = \frac{1}{x^r}$

Theorem 205. $\forall_{x>0} \forall_r \forall_s\ x^r x^s = x^{r+s}$ $[\forall_{x\geq 0} \forall_{r\geq 0} \forall_{s\geq 0}\ x^r x^s = x^{r+s}]$

Theorem 206. $\forall_{x>0} \forall_r \forall_s\ \frac{x^r}{x^s} = x^{r-s}$

Theorem 207. $\forall_{x>0} \forall_r \forall_s\ (x^r)^s = x^{rs}$ $[\forall_{x\geq 0} \forall_{y\geq 0} \forall_{s\geq 0}\ (x^r)^s = x^{rs}]$

Theorem 208. $\forall_{x>0} \forall_{y>0} \forall_r\ (xy)^r = x^r y^r$ $[\forall_{x\geq 0} \forall_{y\geq 0} \forall_{r\geq 0}\ (xy)^r = x^r y^r]$

Theorem 209. $\forall_{x>0} \forall_{y>0} \left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}$ $[\forall_{x\geq 0} \forall_{y>0} \forall_{r\geq 0} \left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}]$

Theorem 210. $\forall_{x>0} \forall_{y>0} \forall_r \left(\frac{x}{y}\right)^{-r} = \left(\frac{y}{x}\right)^r$

Theorem 211. $\forall_{x>0} \forall_r \forall_s [x^r = x^s \Rightarrow (x = 1 \text{ or } r = s)]$

Here are proofs:

Theorem 203. Suppose that $rm \in I$. Then, for $a > 0$, by (3), $a^r = (\sqrt[m]{a})^{rm}$. By Theorem 190b, for $a \geq 0$, $\sqrt[m]{a} > 0$ and, so, by Theorem 152a, since $rm \in I$, $(\sqrt[m]{a})^{rm} > 0$. Consequently, $a^r > 0$.

[The bracketed form of Theorem 193 follows from that just proved together with (3′) and the fact that $1 > 0$.]

Theorem 204. Suppose that $rm \in I$. Then, since I is closed with respect to oppositing [and because of Theorem 21], $-rm \in I$. So [by (R)], $-r \in R$ and, by (3), for $a > 0$,

$$\left.\begin{aligned} a^{-r} &= (\sqrt[m]{a})^{-rm} \\ &= \frac{1}{(\sqrt[m]{a})^{rm}} \end{aligned}\right\} \text{Theorems 154 and 190b}$$
$$= \frac{1}{a^r}.$$

Theorem 205. Suppose that $rm \in I$ and $sn \in I$. Since I is closed with respect to multiplication [and because of the apm and the cpm], $r(mn) \in I$ and $s(mn) \in I$. Since I is closed with respect to addition [and because of the dpma], $(r + s)mn \in I$. Since I^+ is closed with respect to multiplication, $mn \in I^+$. So [by (R)], $r + s \in R$ and, for $a > 0$,

$$\left.\begin{aligned} a^r a^s &= (\sqrt[mn]{a})^{r(mn)} (\sqrt[mn]{a})^{s(mn)} \\ &= (\sqrt[mn]{a})^{(r+s)mn} \end{aligned}\right\} \begin{array}{c}\text{Theorems 155}\\ \text{and 190b}\end{array}$$
$$= a^{r+s}.$$

[The bracketed form of the theorem follows from that just proved together with:

$$\forall_{r \geq 0} \forall_{s \geq 0}\ 0^r 0^s = 0^{r+s}$$

This last is proved, using (3′), by considering two cases, ($r > 0$ or $s > 0$), $r = 0 = s$. In the first case, $0^r 0^s = 0 = 0^{r+s}$; in the second, $0^r 0^s = 1 = 0^{r+s}$.]

Theorem 206. [This follows using Theorems 205 and 203 in the same way in which Theorem 156 follows using Theorems 155 and 152b. Here's how:]

Since R is closed with respect to subtraction, it follows, using Theorem 205, that, for $a > 0$, $a^{r-s}a^s = a^{r-s+s} = a^r$. Since, using Theorem 203, for $a > 0$, $a^s \neq 0$, it follows that, for $a > 0$, $a^{r-s} = a^r/a^s$.

Theorem 207. Suppose that $rm \in I$ and $sn \in I$. For $a > 0$, by Theorem 203, $a^r > 0$. So, by (3), for $a > 0$,

$$\begin{aligned}
(a^r)^s &= \left(\sqrt[n]{a^r}\right)^{sn} \\
&= \left(\sqrt[n]{(\sqrt[m]{a}\,)^{rm}}\right)^{sn} \\
&= \left(\left(\sqrt[n]{\sqrt[m]{a}}\right)^{rm}\right)^{sn} && \text{Theorems 191d, 190b} \\
&= \left(\left(\sqrt[mn]{a}\right)^{rm}\right)^{sn} && \text{Theorem 191c} \\
&= \left(\sqrt[mn]{a}\right)^{(rm)(sn)} && \text{Theorems 157, 190b} \\
&= \left(\sqrt[mn]{a}\right)^{(rs)(mn)} .
\end{aligned}$$

Since $rm \in I$, $sn \in I$, and I is closed with respect to multiplication, it follows that $(rs)(mn) \in I$. Since I^+ is closed with respect to multiplication, $mn \in I^+$. So [by (R)], $rs \in R$ and, by (3), for $a > 0$,

$$(a^r)^s = a^{rs} .$$

[The bracketed form of the theorem follows from that just proved together with:

$$\forall_{r \geq 0} \forall_{s \geq 0}\ (0^r)^s = 0^{rs}$$

This last is proved, using (3′), by considering two cases, ($r = 0$ or $s = 0$), $r > 0 < s$. In the first case, $(0^r)^s = 1 = 0^{rs}$; in the second $(0^r)^s = 0 = 0^{rs}$.]

Theorem 208. Suppose that $rm \in I$. Since, for $a > 0$ and $b > 0$, $ab > 0$, it follows, for $a > 0$ and $b > 0$, that

$$(ab)^r \underset{\substack{\uparrow \\ (3)}}{=} (\sqrt[m]{ab})^{rm} \underset{\substack{\uparrow \\ \text{Th. 191a}}}{=} (\sqrt[m]{a}\,\sqrt[m]{b})^{rm} \underset{\substack{\uparrow \\ \text{Th. 158} \\ \text{Th. 190b}}}{=} (\sqrt[m]{a})^{rm}(\sqrt[m]{b})^{rm} \underset{\substack{\uparrow \\ (3)}}{=} a^r b^r.$$

[The bracketed form of the theorem follows from that just proved together with:

$$\forall_{y \geq 0} \forall_{r \geq 0}\, (0y)^r = 0^r y^r \quad \text{and:} \quad \forall_{x \geq 0} \forall_{r \geq 0}\, (x0)^r = x^r 0^r$$

The first follows from the second and the cpm. The second follows from the pm0 and '$\forall_{x > 0} \forall_{r > 0}\; 0^r = x^r 0^r$'. This last is proved, using (3′), by considering two cases, $r > 0$, $r = 0$. In the first case, $0^r = 0 = a^r 0 = a^r 0^r$; in the second, $0^r = 1 = 1 \cdot 1 = a^r 0^r$.]

Theorem 209. [This follows using Theorems 208 and 203 in the same way in which Theorem 159 follows using Theorems 158 and 152b. Here's how:]

For $a > 0$ and $b > 0$, it follows, using Theorem 208, that

$$\left(\frac{a}{b}\right)^r b^r = \left(\frac{a}{b} \cdot b\right)^r = a^r.$$

Since, using Theorem 203, for $b > 0$, $b^r \neq 0$, it follows that, for $a > 0$ and $b > 0$, $(a/b)^r = a^r/b^r$.

[The bracketed form of the theorem is proved in exactly the same way, using the bracketed form of Theorem 208.]

Theorem 210. For $a > 0$ and $b > 0$, $a/b > 0$ and, using Theorem 203, $a^r \neq 0$ and $b^r \neq 0$. So, for $a > 0$ and $b > 0$,

$$\left(\frac{a}{b}\right)^{-r} \underset{\substack{\uparrow \\ \text{Th. 204}}}{=} \frac{1}{(a/b)^r} \underset{\substack{\uparrow \\ \text{Th. 208}}}{=} \frac{1}{a^r/b^r} \underset{\substack{\uparrow \\ \text{Th. 73}}}{=} \frac{b^r}{a^r} \underset{\substack{\uparrow \\ \text{Th. 209}}}{=} \left(\frac{b}{a}\right)^r.$$

Theorem 211. Suppose, for $a > 0$, that $a^r = a^s$, where $rm \in I$ and $sn \in I$. Then, by [closure properties and] (3), $\left(\sqrt[mn]{a}\right)^{r(mn)} = \left(\sqrt[mn]{a}\right)^{s(mn)}$ and, by Theorem 161, either $\sqrt[mn]{a} = 1$ or $r(mn) = s(mn)$--that is, either $a = 1$ or $r = s$.

EXERCISES

A. For each of the powers listed below, write a simple expression which contains neither an exponent symbol nor a radical.

Sample. $49^{\frac{3}{2}}$

Solution. $49^{\frac{3}{2}} = (\sqrt{49})^3 = 7^3 = 343$

1. $16^{\frac{4}{2}}$
2. $9^{\frac{3}{2}}$
3. $16^{\frac{3}{4}}$
4. $16^{\frac{2}{4}}$
5. $27^{\frac{4}{3}}$
6. $1024^{0.4}$
7. $243^{0.6}$
8. $1024^{\frac{1}{10}}$
9. $1024^{0.3}$
10. $52^{\frac{2}{2}}$
11. $216^{\frac{5}{3}}$
12. $81^{\frac{3}{3}}$
13. $10000^{\frac{1}{2}}$
14. $1000000^{\frac{2}{6}}$
15. $1000000^{\frac{8}{6}}$
16. $\left(81^{\frac{1}{4}}\right)^2$
17. $\left(81^2\right)^{\frac{1}{4}}$
18. $256^{1.25}$
19. $(0.008)^{1.6\bar{6}}$
20. $\left(625^{\frac{2}{5}}\right)^{\frac{5}{4}}$
21. $\left(625^{\frac{5}{4}}\right)^{\frac{2}{5}}$
22. $343^{-0.\bar{3}}$
23. $\left(10^{-6}\right)^{\frac{1}{3}}$
24. $216^{\frac{2}{3}}$
25. $(0.0001)^{-\frac{1}{4}}$
26. $\left(10^{-4}\right)^{-\frac{1}{2}}$
27. $216^{-\frac{1}{3}}$
28. $(16 \cdot 81)^{\frac{3}{4}}$
29. $(81 \cdot 16)^{-\frac{3}{4}}$
30. $\left(\frac{1}{81} \cdot \frac{1}{16}\right)^{-\frac{3}{4}}$

B. Simplify each of the following.

Sample. $$\frac{5^{-\frac{1}{2}} \cdot 5^{\frac{1}{3}}}{5^{\frac{2}{5}} \cdot 5^{-\frac{3}{2}}}$$

Solution. $$\frac{5^{-\frac{1}{2}} \cdot 5^{\frac{1}{3}}}{5^{\frac{2}{5}} \cdot 5^{-\frac{3}{2}}} = 5^{-\frac{1}{2} + \frac{1}{3} - \frac{2}{5} + \frac{3}{2}} = 5^{\frac{14}{15}}$$

1. $3^{\frac{2}{3}} \cdot 3^{-\frac{1}{5}} \cdot 3^{\frac{4}{5}}$

2. $7^{-\frac{1}{3}} \cdot 7^{\frac{1}{5}} \cdot 7^{\frac{1}{2}}$

3. $\dfrac{\pi^{\frac{1}{2}} \cdot \pi^{-\frac{2}{3}}}{\pi^{\frac{1}{5}} \cdot \pi^{\frac{1}{8}}}$

4. $\dfrac{1^{\frac{9}{11}} \cdot 1^{\frac{5}{17}} \cdot 1^{\frac{18}{37}}}{1^{-\frac{2}{5}} \cdot 1^{\frac{7}{8}} \cdot 1^{-\frac{101}{102}}}$

5. $\dfrac{2^{\frac{1}{2}} \times 2^{-\frac{2}{3}}}{2^{\frac{1}{2}} + 2^{-\frac{2}{3}}}$

6. $\dfrac{3^{\frac{1}{2}} + 3^{-\frac{2}{3}}}{3^{\frac{1}{2}} \times 3^{-\frac{2}{3}}}$

7. $\left(16\pi^{8}\right)^{\frac{1}{2}}$

8. $\left(4\pi^{4}\right)^{\frac{1}{2}}$

9. $\left(64\pi^{12}\right)^{\frac{1}{3}}$

10. $\frac{1}{2}\pi^{\frac{1}{2}} \div \left(\frac{1}{3}\pi^{\frac{1}{3}}\right)$

11. $\frac{1}{5}\pi^{\frac{1}{5}} \cdot \frac{1}{6}\pi^{\frac{1}{6}}$

12. $\left(\frac{1}{64}\pi^{\frac{1}{2}}\right)^{\frac{1}{3}}$

13. $\left(\frac{1}{8}\pi^{\frac{1}{2}}\right)^{2}$

14. $\left(-\frac{1}{8}\pi^{-\frac{1}{2}}\right)^{2}$

15. $\left(-\frac{1}{8}\pi^{-\frac{1}{2}}\right)^{-2}$

16. $\dfrac{27^{-\frac{2}{3}} + 27^{\frac{2}{3}}}{3 \cdot 27^{0} + \frac{3}{2} \cdot 27}$

C. Simplify to a form which contains neither fraction exponent symbols nor exponent symbols with oppositing signs. [Assume that the variables have positive values only.]

Sample 1. $$\left(x^{2}y^{6}\right)^{\frac{1}{3}}$$

Solution. $$\left(x^{2}y^{6}\right)^{\frac{1}{3}} = \left(x^{2}\right)^{\frac{1}{3}} \cdot \left(y^{6}\right)^{\frac{1}{3}} = x^{\frac{2}{3}} \cdot y^{2} = \left(\sqrt[3]{x}\right)^{2} y^{2}$$

Sample 2. $\left(x^2\right)^{\frac{7}{10}} + 5x^{\frac{7}{5}}$

Solution. $\left(x^2\right)^{\frac{7}{10}} + 5x^{\frac{7}{5}} = x^{\frac{14}{10}} + 5x^{\frac{7}{5}}$

$= x^{\frac{7}{5}} + 5x^{\frac{7}{5}}$

$= 6x^{\frac{7}{5}}$

$= 6x\left(\sqrt[5]{x}\right)^2$

1. $\left(x^{\frac{1}{2}}y^{\frac{3}{2}}\right)^2$

2. $\left(a^3b^9\right)^{\frac{2}{3}}$

3. $\left(8a^{\frac{3}{5}}c^6\right)^{\frac{2}{3}}$

4. $\left(4 \cdot \sqrt[4]{b} \cdot \sqrt[3]{c}\right)\left(2b^{\frac{3}{4}}c^{\frac{5}{3}}\right)$

5. $\left(81p^8\right)^{0.75}$

6. $\left(x^3 + y^6\right)^{\frac{1}{6}}$

7. $\left(x^{\frac{1}{2}}\right)^3 + x^{\frac{1}{2}}$

8. $\left(x^{\frac{1}{2}}\right)^3 + \left(x^3\right)^{\frac{1}{2}}$

9. $\left(\dfrac{16a^3t^6}{9a^3}\right)^{\frac{3}{2}}$

10. $\left(4a^2x^4\right)^{\frac{3}{2}} + \left(a^6x^{12}\right)^{\frac{1}{2}}$

11. $\left(27a^{-3}b^{-12}\right)^{\frac{2}{3}}$

12. $\left(36x^{-4}y^6z^{-8}\right)^{-\frac{3}{2}}$

13. $4^{-\frac{1}{2}}$

14. $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

15. $64^{-\frac{2}{3}}$

16. $8^{-2/3}$

17. $8^{-4/3}$

18. $64^{-1/4}$

19. $0^{\frac{2}{3}}$

20. $(-0)^{\frac{2}{3}}$

21. $0^{-\frac{2}{3}}$

22. $243^{-3/5}$

23. $243^{3/-5}$

24. $81^{-0.25}$

25. $\left(125x^6\right)^{-\frac{1}{3}}$

26. $\left(125x^6\right)^{-\frac{2}{3}}$

27. $\left[\left(132x^4a^3\right)^{\frac{5}{7}}\right]^0$

28. $\left(x^{-2}\right)^{-\frac{1}{2}}$

29. $\left(27x^3\right)^{-\frac{1}{3}}$

30. $\left(64c^{12}\right)^{-\frac{2}{3}}$

31. $\left(\dfrac{1}{a^{-2}}\right)^{-4}$

32. $\left(\dfrac{b^{-3}}{a^{-2}}\right)^{-\frac{1}{6}}$

33. $\left(\dfrac{a^3b^{-3}}{ac^{-4}}\right)^{-\frac{1}{3}}$

34. $\left(a^{\frac{1}{2}} b^{\frac{2}{5}} a^{\frac{1}{3}} b^{\frac{1}{2}} a^{-\frac{1}{4}}\right)^{-4}$

35. $\left[\left(x^{-m}\right)^{\frac{1}{n}}\right]^{-n/m^2}$

36. $\left[\dfrac{x^{m/n}}{x^{n/m}}\right]^{\frac{1}{mn}}$

37. $\left[\dfrac{x^{2n+1}y^{-n+4}}{x^{-3n+2}y^{-2n+3}} \cdot \dfrac{x}{y}\right]^{\frac{1}{n}}$

38. $\left(4ac^2\right)^{\frac{1}{3}} \cdot \left(9b^4c^5\right)^{\frac{1}{3}} \cdot \left(6a^2b^2c^2\right)^{\frac{1}{3}}$

D. True or false?

1. $5^0 = 0$

2. $6^{-5} \cdot 6^5 = 1$

3. $7^{-3} \cdot 7^{-5} = 7^{15}$

4. $1^{-3} \cdot 1^{-5} = 1^{15}$

5. $-5^2 = -25$

6. $(-5)^2 = 25$

7. $\left(3^5\right)^2 = 3^{5^2}$

8. $9^3 \div 9^5 = 9^{-5} \cdot 9^3$

9. $\dfrac{1}{3^4 + 2^5} = 3^{-4} + 2^{-5}$

10. $(\pi + 7)^{-3} = \pi^{-3} + 7^{-3}$

11. $4^{-2} = -8$

12. $\left(\dfrac{6\pi^{-3}4^{-2}}{5^{-7}}\right)^0 = 5^0$

13. $\dfrac{\pi^{-1}}{7^{-1}} = \dfrac{7}{\pi}$

14. $-2^{-1} = -\frac{1}{2}$

15. $(-3)^{-2} = \frac{1}{9}$

16. $-3^{-2} = -\frac{1}{9}$

17. $5^0 \times 5^3 = 5^0$

18. $2^3 \times 2^5 = 4^8$

19. $0^1 + 0^2 = 0^3$

20. $1^5 \times 1^2 = 1^{10}$

21. $\pi^{7-3} = \dfrac{1}{\pi^{3-7}}$

22. $3^1 + 3^{-1} = 10\left(3^{-1}\right)$

23. $0^5 = 1^5$

24. $0^0 = 0^1$

25. $\forall_{x \geq 0} \forall_{y \geq 0} \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3 = x + y$

26. $\forall_{x \geq 0}\ x^{\frac{1}{2}} = \frac{x}{2}$

27. $3 = 27^{\frac{1}{9}}$

28. $-4 = 16^{\frac{1}{2}}$

29. $\left(\frac{1}{3}\right)^{-3} = \frac{1}{27}$

30. $\left(\frac{\pi}{\sqrt{2}}\right)^{-5} = \left(\frac{\sqrt{2}}{\pi}\right)^{5}$

31. $2^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$

32. $\pi^2 + \pi^3 = \pi^5$

33. $\left[\left(\frac{1}{3}\right)^{-\frac{1}{7}}\right]^0 = 1$

34. $\dfrac{\pi^{-3}}{(\sqrt{5})^{-4}} = \dfrac{5^2}{\pi^3}$

35. $\sqrt[3]{4^4} = 4^{\frac{3}{4}}$

36. $7^{-\frac{1}{2}} \times 7^{\frac{1}{2}} = 1$

37. $0^{\frac{1}{2}} = 0$

38. $(\pi + \sqrt{2})^{-3} \div (\pi + \sqrt{2})^3 = 1$

39. $10^{0.3} = \left(\frac{1}{10}\right)^3$

40. $\pi^{-\frac{1}{3}}\left(\pi^{\frac{4}{3}} + \pi^0\right) = \pi + \dfrac{1}{\sqrt[3]{\pi}}$

41. $10^{0.3} = 5^{0.6}$

42. $(0.25)^{-0.5} = 2$

E. Expand. [Assume that the values of the variables are positive numbers only.]

Sample. $\left(x^{\frac{1}{2}} + y^2\right)^2$

Solution. $\left(x^{\frac{1}{2}} + y^2\right)^2 = x + 2y^2 x^{\frac{1}{2}} + y^4$

1. $\left(m^{\frac{1}{2}} + n\right)^2$

2. $\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2$

3. $\left(x^{-\frac{1}{2}} + y^{-\frac{1}{2}}\right)^2$

4. $\left(b^{\frac{1}{2}} + c^{\frac{1}{2}}\right)\left(b^{\frac{1}{2}} - c^{\frac{1}{2}}\right)$

5. $\left(a^{\frac{3}{2}} + 2a^{\frac{1}{2}} - 4a^{-\frac{1}{2}}\right)\left(2a^{\frac{3}{2}} - 5a^{\frac{1}{2}}\right)$

6. $\left(2a^{\frac{5}{2}} - 3a^{\frac{3}{2}} + 4a^{\frac{1}{2}}\right)\left(7a^{\frac{1}{2}} - 3a^{-\frac{1}{2}}\right)$

F. Find the rational approximation correct to the nearest tenth.

Sample. $10^{\frac{1}{4}}$

Solution. Since $1^4 < 10 < 2^4$, $1 < 10^{1/4} < 2$. So, a first approximation might be 1.5. We check this by computing:

$$(1.5)^4 = \left[(1.5)^2\right]^2 = (2.25)^2 < 5.1$$

Since $5.1 < 10$, $1.5 < 10^{1/4}$. Our first approximation is too small. Try 1.8, and compute again:

$$(1.8)^4 = (3.24)^2 > 10.4$$

So, 1.8 is a bit too large. Try 1.7.

$$(1.7)^4 = (2.89)^2 < 8.4.$$

So, $1.7 < 10^{1/4} < 1.8$. Since we are looking for the rational approximation to $10^{1/4}$ correct to the nearest tenth, all we need to do is to decide between 1.7 and 1.8. One more calculation will tell us:

$$(1.75)^4 = (3.0625)^2 < 10$$

So, 1.8 is the rational approximation correct to the nearest tenth.

1. $10^{\frac{1}{3}}$ 2. $5^{3/4}$ 3. $2^{\frac{2}{5}}$ 4. $4^{3.5}$

RATIONAL EXPONENTS AND NEGATIVE BASES

The definition which we have been using of rational powers of positive numbers:

(3) $$\forall_{x>0} \forall_r \forall_{m,\ rm \in I}\ x^r = (\sqrt[m]{x})^{rm}$$

can be supplemented by a definition of certain rational powers of negative numbers. For example, ignoring the restriction '$x > 0$', we can use (3) to make sense of '$(-8)^{2/3}$':

$$(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$$

In a similar manner, for any rational number r for which there is an odd positive integer m with $rm \in I$, and for any $a \neq 0$, we can agree that

$$a^r = (\sqrt[m]{a})^{rm}.$$

[If $a > 0$, this is merely a consequence of (3). The case in which $a = 0$ is covered by (3′).]

A check of the proofs of Theorems 204-210 shows that the assertions made in these theorems for positive bases continue to hold for nonzero bases provided that the exponents are of this special kind. Although Theorem 203 cannot be generalized to this extent $[(-8)^{1/3} = -2 \not> 0]$, its role in proving the later theorems is served as well by:

$$\forall_{x>0} \forall_r \; x^r \neq 0$$

and this does generalize to nonzero bases if the exponents are restricted as mentioned. Theorems 154-160, which are used in the proofs, are stated for nonzero bases; and the parts of Theorem 191 are paralleled by those of Theorem 191′ [See page 9-359 of the theorem list.]. The role of Theorem 190 can be played by the readily proved theorem '$\forall_{x\neq 0} \forall_{m \in O} \sqrt[m]{x} \neq 0$' [see Theorems 190 and 190′].

The upshot of all this is that as long as you are dealing with nonzero bases you can perform the usual manipulations provided that the exponents are rational numbers which have odd multiples belonging to I.

EXERCISES

A. Which of the following expressions make sense?

Sample 1. $(-8)^{\frac{4}{6}}$

Solution. This makes sense because there is an odd positive integer m [to wit, 3] such that $\frac{4}{6}m \in I$. So, $(-8)^{\frac{4}{6}} = \left(\sqrt[3]{-8}\right)^{\frac{4}{6}\cdot 3} = 4$. Note, however, that it makes no sense to say: $(-8)^{\frac{4}{6}} = \left(\sqrt[6]{-8}\right)^4$

Sample 2. $(-8)^{\frac{7}{12}}$

Solution. This does not make sense because there is no odd positive integer m such that $\frac{7}{12}m \in I$. [How do you know there isn't?]

1. $(-8)^{\frac{3}{18}}$ 2. $(-9)^{\frac{16}{24}}$ 3. $(-12)^{\frac{100}{50}}$ 4. $(-2)^{\frac{21}{28}}$

B. 1. On the same chart, sketch the graphs of the equations:

$$y = x^{\frac{1}{3}}, \quad y = x^{\frac{2}{3}}, \quad y = x^{\frac{3}{3}}, \quad \text{and:} \quad y = x^{\frac{4}{3}}$$

2. Repeat Exercise 1 for the graphs of the equations:

$$y = x^{\frac{1}{4}}, \quad y = x^{\frac{2}{4}}, \quad y = x^{\frac{3}{4}}, \quad \text{and:} \quad y = x^{\frac{5}{4}}$$

EXPLORATION EXERCISES

A. You know that $2^0 = 1$. Also, $2^{1/4} \doteq 1.1892$, $2^{1/2} \doteq 1.4142$, and $2^{3/4} \doteq 1.6818$. Use these data to make a table of powers of 2 from 2^{-4} to 2^4, changing the exponent by 1/4 from one entry to the next. [Hint. $2^{5/4} = 2^{1/4} \cdot 2^1$; $2^{-1/4} = 2^{3/4} \div 2^1$. Your table should have 33 entries.]

B. Round off the entries in your table to two decimal places, and use the results to plot points on a graph of the exponential function with base 2 and with rational arguments, from −4 to 4. [Hint. Use as large a scale as your paper allows, and a sharp pencil. Do as accurate a job as you can.]

C. Draw as smooth a curve as you can through the points you plotted in Part B. [Hint. Turn your paper more or less topside down, and use your elbow as a pivot.] Do you think that, if you plotted other pairs $(r, 2^r)$, for rational numbers between −4 and 4, these points would lie on your curve?

D. Use your graph to find an approximate solution for each of the following equations.

Sample 1. $2^{1.4} = x$

Solution. The ordinate of the point of the graph whose abscissa is 1.4 is approximately 2.6. So, the solution is approximately 2.6.

[Check: $2^{1.5} = 2\sqrt{2} \doteq 2 \cdot 1.4 = 2.8$]

Sample 2. $2^x = 10$

Solution. The abscissa of the point of the graph whose ordinate is 10 is approximately 3.3. So, the solution is approximately 3.3.

[Check: $2^3 = 8$]

Sample 3. $4^x = 6$

Solution. Since $4 = 2^2$, $4^x = 6$ if and only if $(2^2)^x = 6$--that is [by Theorem 197], if and only if $2^{2x} = 6$. From the graph, $2^{2.6} \doteq 6$. So, the solution is approximately 1.3.

1. $2^{2.3} = x$	2. $2^x = 5$	3. $2^{-0.3} = x$
4. $2^x = 3.5$	5. $2^{2x-3} = 3.5$	6. $2^{3x+4} = -2$
7. $8^{2x} = 4.2$	8. $4^x = 2^{x-2}$	9. $2^{\sqrt{2}} = x$

E. 1. Your work with the exponential function with base 2 and rational arguments has probably suggested to you that it is an increasing function--that is, that

$$\forall_{r_1} \forall_{r_2} [r_2 > r_1 \Rightarrow 2^{r_2} > 2^{r_1}].$$

Here is an outline of a proof that this is the case. Your job is to

fill in the blanks.

Since $2 > 0$ and R is closed with respect to ___________, it follows by Theorem 20_, that $2^{r_2} = 2^{r_2 - r_1} \cdot 2^{r_1}$. So, since $2^{r_1} = 1 \cdot 2^{r_1}$, it follows that

$$2^{r_2} > 2^{r_1} \text{ if and only if } \_\_\_\_\_ \cdot 2^{r_1} > \_\_ \cdot 2^{r_1}.$$

Since, by Theorem 20_, 2^{r_1} __ __, it follows that

$$2^{r_2 - r_1} \cdot 2^{r_1} > 1 \cdot 2^{r_1} \text{ if and only if } 2^{r_2 - r_1} > 1.$$

Suppose, now, that $r_2 > r_1$. Since R is closed with respect to subtraction, there is, by (R), an m such that $(r_2 - r_1)m$ _ __. Since $r_2 - r_1$ is _________, it follows that $(r_2 - r_1)m$ is some positive integer--say, p. By definition (3) on page 9-72, it follows that

$$2^{r_2 - r_1} = \_\_\_\_\_\_, \text{ and, of course, } 1 = (\sqrt[m]{1})^p.$$

Hence,

$$2^{r_2 - r_1} > 1 \text{ if and only if } \_\_\_\_\_ > \_\_\_\_\_.$$

Consequently, assuming that _____,

$$2^{r_2} > 2^{r_1} \text{ if and only if } (\sqrt[m]{2})^p > (\sqrt[m]{1})^p.$$

Now, since $2 > 1$ and since the mth principal root function is an ___________ function [Theorem 189], it follows that $\sqrt[m]{2}$ ___ $\sqrt[m]{1}$. From this, by (PR) and the fact that the pth _________ __________ restricted to nonnegative arguments is __________, it follows that _____ > _____. Hence, ___ > ___.

Consequently, if $r_2 > r_1$ then $2^{r_2} > 2^{r_1}$.

2. Complete:

(a) For a __ 1, the exponential function with base a and rational arguments is increasing.

(b) For ___ ___ a ___ ___, the exponential function with base a and rational arguments is decreasing.

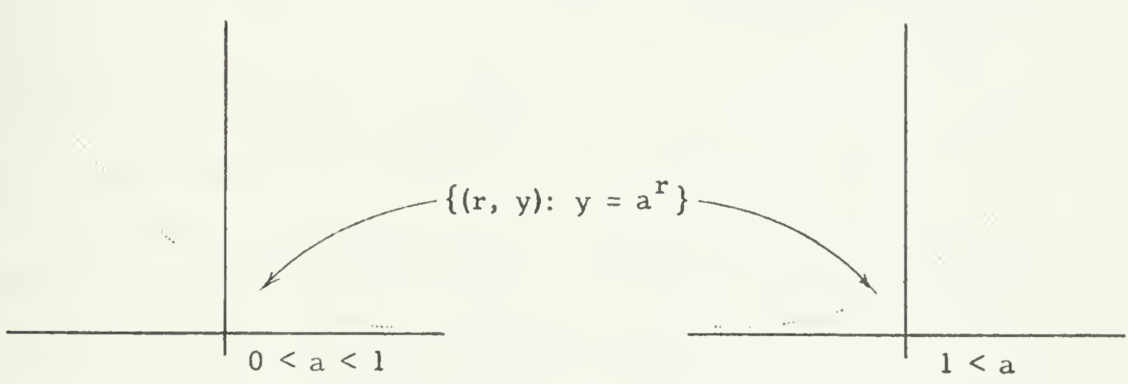
9.07. The exponential functions.--In the preceding section we extended the domain of the exponential functions with positive bases to include all rational numbers, and we proved a number of important theorems [Theorems 203-211 on page 9-75]. In this section we shall complete the definition of these functions by indicating how to interpret irrational exponents.

MORE ABOUT RATIONAL EXPONENTS

The last numbered theorem of the preceding section was Theorem 211 for rational exponents:

$$\forall_{x>0} \forall_r \forall_s \, [x^r = x^s \Longrightarrow (x = 1 \text{ or } r = s)]$$

This theorem tells us that, for $0 < a \neq 1$, the exponential function with base a and rational arguments has an inverse [Explain.]. Since most of the functions we have met which have inverses are also monotonic, Theorem 211 suggests that these functions may be monotonic. Also, work with the one with base 2, in Exercise 1 of Part E on pages 9-87 and 9-88, bears out this conjecture. The conjecture is verified in Appendix C [pages 9-250 through 9-269].



Your work with the exponential function with base 2 and rational arguments may also have suggested that exponential functions with positive bases and rational arguments are continuous--that is, sufficiently small changes in argument result in arbitrarily small changes in value. This conjecture is also correct, and is verified in Appendix C.

Finally, Theorem 203 tells us that the range of each exponential function with positive base and rational arguments is a subset of

$\{x: x > 0\}$. [If you have studied Appendix B you will know that the range must, then, be a proper subset of $\{x: x > 0\}$.] Your work with base 2 probably suggests that [as long as the base is not 1] an exponential function has arbitrarily large values. Consequently [Theorem 204], such a function has arbitrarily small positive values.

Collecting these results, we have:

> For $0 < a \neq 1$, the exponential function with base a and rational arguments is a continuous monotonic function--decreasing if $0 < a < 1$ and increasing if $a > 1$.
>
> For $0 < a < 1$, given $M > 0$, there is an N such that, for each $r > N$, $a^{-r} > M$ and $0 < a^r < 1/M$.
>
> For $a > 1$, given $M > 0$, there is an N such that, for each $r > N$, $a^r > M$ and $0 < a^{-r} < 1/M$.

The proof of monotonicity is like that given for the case $a = 2$ in Part E on page 9-88. The proof of continuity makes use of Bernoulli's Inequality [Theorem 162]. The last two results also depend on Bernoulli's Inequality. For details, see Appendix C.

IRRATIONAL EXPONENTS

How should we define '$2^{\sqrt{2}}$'? As with any definition, we are free to choose--but, we shall be stuck with whatever consequences follow from our choice. In defining '2^{-3}', say, and '$2^{3/4}$', we attempted [successfully] to control the consequences by choosing definitions which would enable us to justify the laws which we already knew to hold for nonnegative integral exponents. Here, similarly, we shall do well to pay attention to what we already know about the exponential function with base 2 and rational exponents.

As it turns out, the simplest property to consider in looking for a definition of '$2^{\sqrt{2}}$' is monotonicity. Monotonic functions are especially simple--in particular, they have inverses--and, as you proved in Part E on page 9-88, the exponential function with base 2 and rational arguments is an increasing function. In extending the definition of the exponential functions to include irrational arguments, we shall try to preserve

this property--in particular, we shall try to define '$2^{\sqrt{2}}$' in such a way that, for each rational number r,

$$\text{(a)} \qquad \text{if } r < \sqrt{2} \text{ then } 2^r < 2^{\sqrt{2}}$$

and

$$\text{(b)} \qquad \text{if } \sqrt{2} < r \text{ then } 2^{\sqrt{2}} < 2^r.$$

It will be of help to restate the requirement (a) in terms of a set.

$$\{y: \exists_{r < \sqrt{2}}\ y = 2^r\}.$$

The requirement (a) is merely that $2^{\sqrt{2}}$ is to be greater than each member of this set--in other words, it is to be an upper bound of the set which does not belong to the set. Now, it is not hard to show that, given any member of the set, there is a greater member--that is, no member of the set is an upper bound of the set. [Proof: Given a rational number $r < \sqrt{2}$ then, by Theorem 201 on page 9-67, there is a rational number s such that $r < s < \sqrt{2}$. Since the exponential function with base 2 and rational arguments is increasing, it follows that $2^s > 2^r$.] So, requirement (a) will be satisfied if and only if $2^{\sqrt{2}}$ is some upper bound of $\{y: \exists_{r < \sqrt{2}}\ y = 2^r\}$.

Now, let's look at requirement (b). Our chance of satisfying this requirement is better the smaller $2^{\sqrt{2}}$ is. So, our best chance of satisfying both (a) and (b) is to agree that

$$2^{\sqrt{2}} = \text{the } \underline{\text{least}} \text{ upper bound of } \{y: \exists_{r < \sqrt{2}}\ y = 2^r\}.$$

However, before adopting this agreement, we had better make sure that $\{y: \exists_{r < \sqrt{2}}\ y = 2^r\}$ does have a least upper bound. According to the ℓubp, all we need do is show that the set in question is nonempty and that it does have some upper bound. This is easy. There is a rational number less than $\sqrt{2}$--for example, the number 0. So, the set is not empty --for example, 2^0 belongs to the set. On the other hand, there is a rational number greater than $\sqrt{2}$--for example, the number 1.5. So, if $r < \sqrt{2}$ then $r < 1.5$. Since the exponential function with base 2 and rational arguments is increasing, it follows that if $r < \sqrt{2}$ then $2^r < 2^{1.5}$. Hence, $2^{1.5}$ is an upper bound of the set. Consequently, the set in question does have a least upper bound.

THE EXPONENTIAL FUNCTIONS

The definitions we have so far adopted:

$$(1)\quad \forall_x \forall_{k \geq 0}\ x^k = \prod_{p=1}^{k} x,$$

$$(2)\quad \forall_{x \neq 0} \forall_{k < 0}\ x^k = \frac{1}{x^{-k}},$$

and: $$(3)\quad \forall_{x>0} \forall_r \forall_{m,\ rm \in I}\ x^r = (\sqrt[m]{x})^{rm}$$

define, for each $x > 0$, the exponential function with base x and rational arguments--that is, $\{(u, y),\ u \in R\colon\ y = x^u\}$. We shall now remove the restriction '$u \in R$', by adopting:

$$(4_1)\quad \forall_{x>1} \forall_u\ x^u = \text{the least upper bound of } \{y\colon \exists_{r<u}\ y = x^r\},$$

$$(4_2)\quad \forall_{0<x<1} \forall_u\ x^u = (1/x)^{-u},$$

and: $(4_3)\quad \forall_u\ 1^u = 1$ and $\forall_{u>0}\ 0^u = 0$ [Of course, by (1), $0^0 = 1$.]

To justify the adoption of these definitions, we must do three things.

First, as to (4_1), we must show that, for any $a > 1$ and any b, $\{y\colon \exists_{r<b}\ y = a^r\}$ does have a least upper bound. [Once this is done, we are sure that (4_1) makes sense. That (4_2) makes sense then follows from Theorem 164; and there is no question about (4_3).]

Second, we must show that the new definitions are consistent with those already adopted. For example, we must show that, for any $a > 1$ and any $b \in R$, the least upper bound of $\{y\colon \exists_{r<b}\ y = a^r\}$ is, if $bp \in I$, $(\sqrt[p]{a})^{bp}$.

Third, we must show that, with these definitions, we can prove Theorems 203-211, for real number exponents.

The first is easy to do. By the ℓubp, all that is needed is to prove that, for any $a > 1$ and any b, $\{y\colon \exists_{r<b}\ y = a^r\}$ is nonempty and has an upper bound. That the set in question is nonempty follows from the fact that, for any b, there is a rational number less than b [Theorem 201, on page 9-67, or the cofinality principle]. That the set has an upper bound follows from the fact that, for $a > 1$, the exponential function with base a and rational exponents is increasing, and the fact that, for any b, there

is a rational number greater than b [again, Theorem 201 or the cofinality principle].

The second and third tasks are somewhat more difficult, and are carried out in Appendix C. For the third, we shall make use of:

> **Theorem 202.**
>
> For each $x > 0$, the exponential function with base x is a continuous function whose domain is the set of all real numbers; if $x \neq 1$, its range is the set of all positive numbers; it is decreasing if $0 < x < 1$ and increasing if $x > 1$.

which is proved in Appendix C.

EXERCISES

A. Use the graph you made in Part C of the Exploration Exercises on page 9-86 to estimate each of the following. [Use Theorems 205 and 207 as suggested in the Hints.]

1. $2^{\sqrt{2}}$ 2. 2^{π} 3. $2^{-\pi/4}$

4. $4^{\sqrt{3}}$ 5. $8^{\sqrt{2}}$ [Hint. $2^{3\sqrt{2}} = 2^{(3\sqrt{2}-1)} \cdot 2$]

6. $3^{\sqrt{3}}$ [Hint. If $2^x = 3$ then $3^{\sqrt{3}} = (2^x)^{\sqrt{3}}$.]

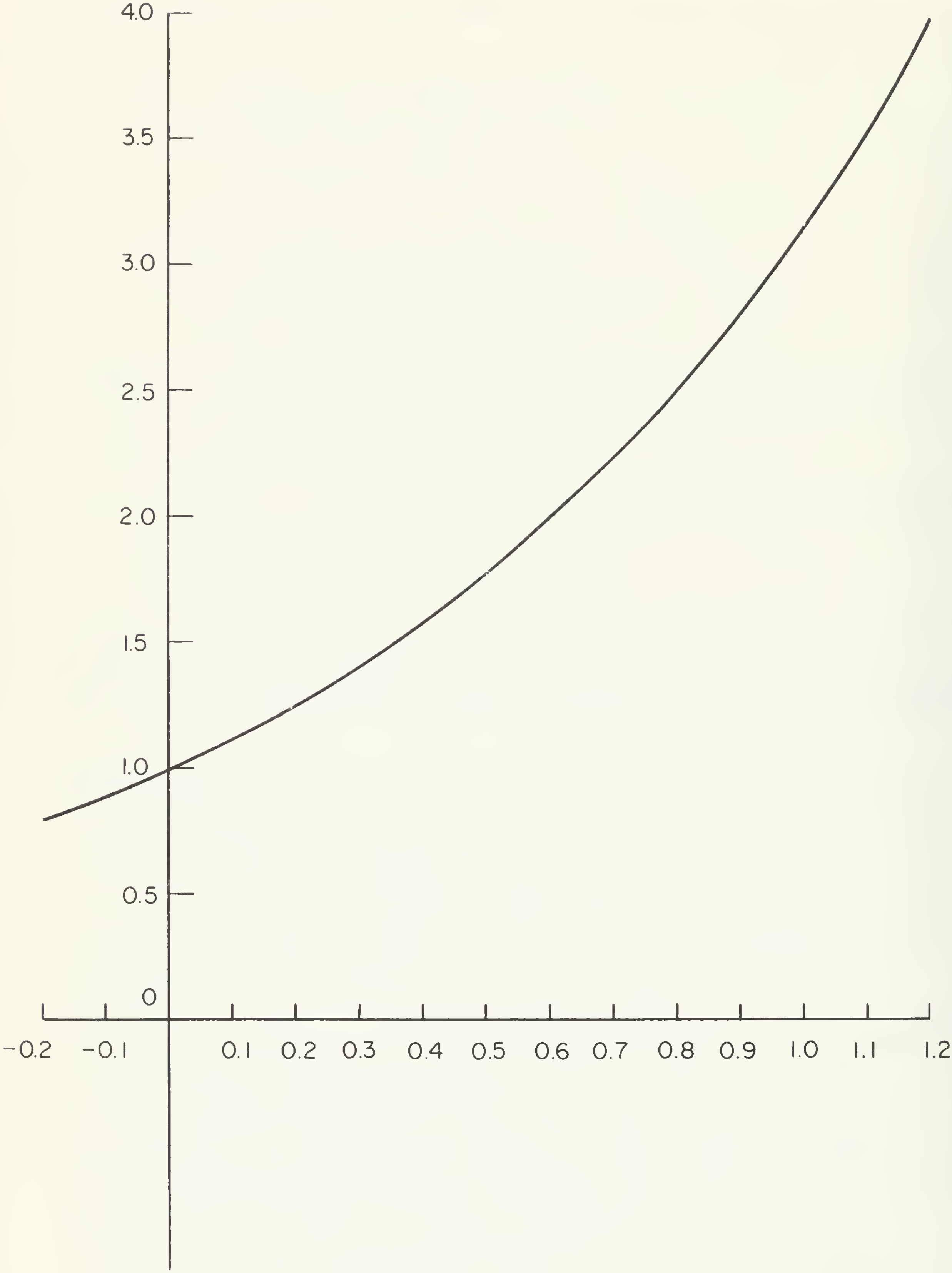
B. True or false? [Use Theorem 202.]

1. $2^{5\sqrt{2}} > 2^{4\sqrt{3}}$ 2. $(\pi/4)^{2\sqrt{3}} > (\pi/4)^{\sqrt{11}}$

3. $3^{\pi\sqrt{3}} > 3 \cdot 3^{\sqrt{3}}$ 4. $(1/\sqrt{2})^{\pi} > (1/\sqrt{2})^{22/7}$

C. Simplify.

1. $3^{\sqrt{2}}/3^{\sqrt{8}}$ 2. $(5^{\pi})^{2/\pi}$ 3. $[(-6)^{\pi}]^{2/\pi}$

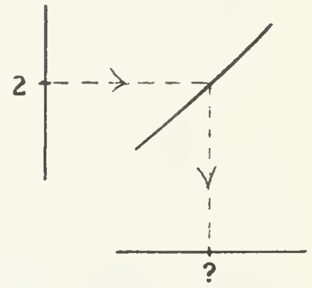

4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
-0.2
-0.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2

EXPLORATION EXERCISES

A. On page 9-94 is a graph of part of the exponential function with base π. Use it to find approximate solutions to the following equations.

Sample. $\pi^x = 2$

Solution. This problem amounts to finding a number x such that (x, 2) belongs to the exponential function with base π. Just find an approximation to the first component of the point whose second component is 2. An approximate solution is 0.61.



1. $\pi^x = 3.14$ 2. $\pi^x = 3$ 3. $\pi^x = 1$ 4. $\pi^x = 2\pi$ 5. $\pi^x = 3.5\pi^2$

* * *

The exponential function with base π has [since $\pi > 1$, and by Theorems 202 and 184] an inverse which is an increasing function whose domain is the set of all positive numbers and whose range is the set of all real numbers. [Explain.]

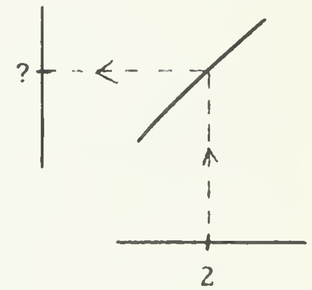
* * *

B. 1. Sketch the inverse of the exponential function with base π.

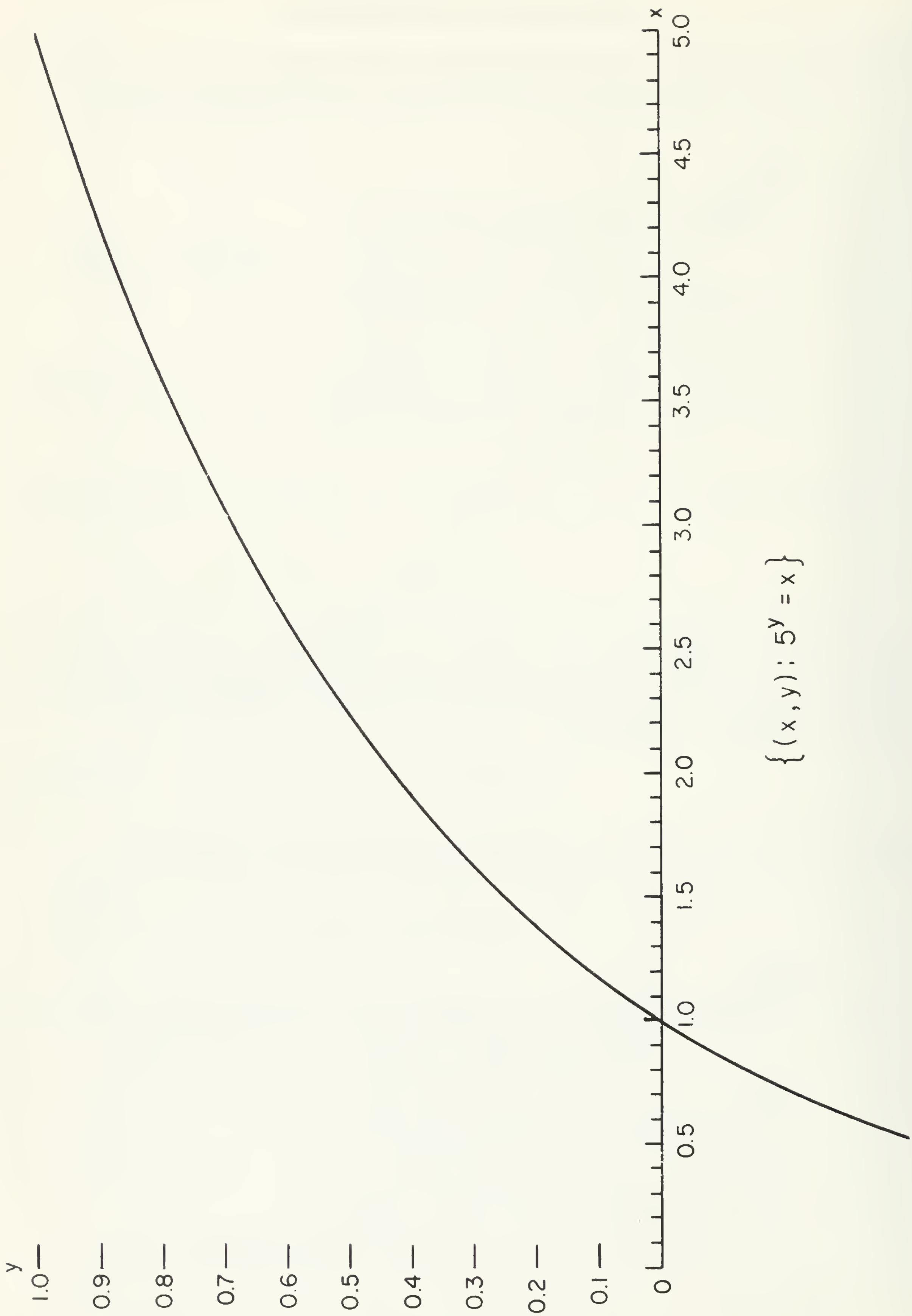
2. Use your sketch to find approximate solutions to these equations.

Sample. $\pi^x = 2$

Solution. This problem amounts to finding a number x such that (2, x) belongs to the inverse of the exponential function with base π. Just find an approximation to the second component of the point whose first component is 2. An approximate solution is 0.61.



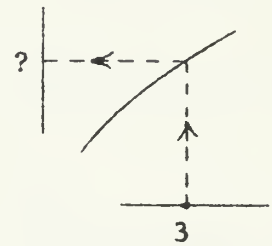
(a) $\pi^x = 3.3$ (b) $\pi^x = 1$ (c) $\pi^x = \sqrt{\pi}$ (d) $\pi^x = 0.8$ (e) $\pi^{3x} = 2.7$


x
y
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\{(x, y) : 5^y = x\}$

C. The chart on page 9-96 shows a graph of part of the inverse of the exponential function with base 5. Use it to help you find approximate solutions to these equations.

Sample. $25^x = 75$

Solution.
$$25^x = 75$$
$$(5^2)^x = 5^2 \cdot 3$$
$$5^{2x}/5^2 = (5^2 \cdot 3)/5^2$$
$$5^{2x-2} = 3$$



From the graph, we see that $2x - 2$ is approximately 0.68. So, a solution is approximately 1.34.

1. $25^x = 15$
2. $125^x = 20$
3. $5^x = -2$
4. $5^{-x} = 10$
5. $25^x = 5^{x-2}$
6. $625^{2x-3} = 18$
7. $5^x = 6250$
8. $5^x = 55.5$
9. $5^x = 5^{-x}$
10. $5^x = 2.6$
11. $5^x = 3.7$
12. $5^x = 2.6 \times 3.7$

* * *

Exercises 10, 11, and 12 suggest an interesting use for the inverse of the exponential function with base 5. Let us suppose that we wish to compute 2.6×3.7. We know that there are numbers a and b such that

$$2.6 = 5^a \quad \text{and} \quad 3.7 = 5^b.$$

So,
$$2.6 \times 3.7 = 5^a \times 5^b = 5^{a+b}.$$

From the chart, $a \doteq 0.59$ and $b \doteq 0.81$. So, $a + b \doteq 1.4$--that is,

$$2.6 \times 3.7 \doteq 5^{1.4}$$
$$= 5^{0.4} \times 5.$$

Now, from the chart,
$$5^{0.4} \doteq 1.9.$$

Hence,
$$2.6 \times 3.7 \doteq 1.9 \times 5 = 9.5.$$

Here is another example of how the chart on page 9-96 can be used to approximate the results of computations. Suppose that we wish to compute $7.2 \div 3.5$. Now, 7.2 is not in the range of arguments covered by the graph. But

$$7.2 = 1.44 \times 5,$$

and, from the graph, $$1.44 \doteq 5^{0.23}.$$

So, $$7.2 \doteq 5^{0.23} \times 5^{1} = 5^{1.23}$$

and, from the graph, $$3.5 \doteq 5^{0.78}.$$

Consequently,

$$\begin{aligned} 7.2 \div 3.5 &\doteq 5^{1.23} \div 5^{0.78} \\ &= 5^{1.23 - 0.78} \\ &= 5^{0.45} \\ &\doteq 2.1. \end{aligned}$$

As a third example, let's use the graph to approximate $(0.39)^2$.

$$\begin{aligned} 0.39 &= 3.0 \times 10^{-1} \\ (0.39)^2 &= (3.9 \times 10^{-1})^2 \\ &= (3.9)^2 \times 10^{-2} \\ &\doteq (5^{0.85})^2 \times 10^{-2} \\ &= 5^{1.7} \times 10^{-2} \\ &= 5^{0.7} \times 5 \times 10^{-2} \\ &\doteq 3.08 \times 5 \times 10^{-2} \\ &= 0.154 \end{aligned}$$

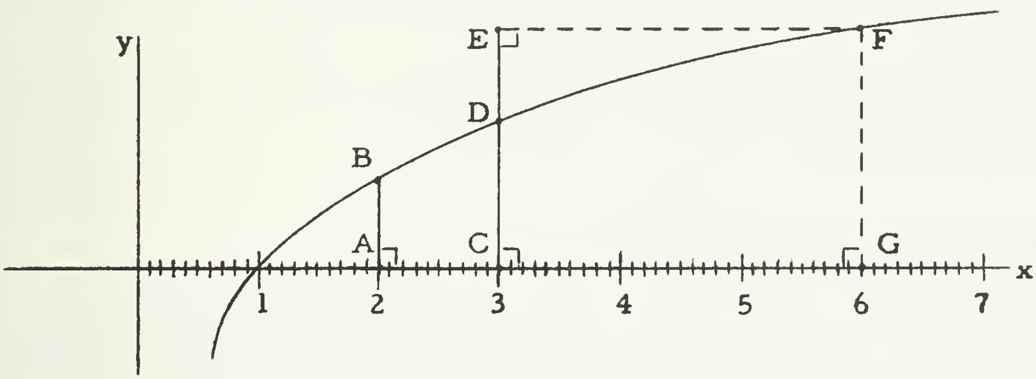
D. Use the graph on page 9-96 to obtain an approxiination to each of the following.

1. 7.2×3.5
2. 0.45×3.5
3. $20.2 \div 6.4$
4. $(9.7)^3$
5. $\sqrt{60}$ [Hint. $60 = ? \times 25$]
6. $40.5 \div 38.6$
7. $\dfrac{12.4 \times 8.5}{22.9}$
8. 2.9π

* * *

In Part D you used a graph of the inverse of the exponential function with base 5 to make computing easier. You can achieve similar results using a graph of the inverse of any exponential function. You can carry out the work directly on the graph without reading off the ordinates.

Here is a graph of the inverse of an exponential function. Let's use it to find an approximation to something simple--say, 2×3. If the base



of the exponential function is b, then

$$2 = b^{AB} \quad \text{and} \quad 3 = b^{CD}.$$

So,

$$2 \cdot 3 = b^{AB} \cdot b^{CD}$$
$$= b^{AB + CD}.$$

If $DE = AB$, then $CE = AB + CD = GF$, where

$$b^{GF} = 6.$$

Hence, $2 \times 3 = 6$.

Do you have to know the base b when using this geometric method for computing?

* * *

E. Use geometric methods and the graph given above to compute each of the following.

1. 1.5×3 2. $5 \div 2$ 3. $2^{3/2}$

4. 24×2.8 [Hint. Since $24 = 2.4 \times 10$, if $2.4 = b^a$ and $2.8 = b^c$ then $24 \times 2.8 = b^{a+c} \times 10$.]

9.08 Computing with inverses of exponential functions. --You have seen how the inverse of an exponential function can be used to replace multiplication, division, and exponentiation steps in computations by additions, subtractions, and multiplications, respectively. Using a graph of one such function--the inverse of the exponential function for the base 5--you have carried out some such computations. The results you obtained were probably not very accurate, but this was to be expected, since the approximations you could read from the graphs to solutions of equations of the form '$x = 5^y$' were not very accurate. If we could improve the accuracy of these approximations, we would expect the accuracy of our results to be improved.

OBTAINING BETTER APPROXIMATIONS

For an example, consider the problem of using the inverse of $\{(x, y): y = 5^x\}$ to find $(19.5)^{7/8}$. Since 19.5 is not one of the arguments covered by the graph on page 9-96, let us begin by dividing by 5-- $19.5 = 3.9 \times 5$. Now, from the graph,

$$3.9 \doteq 5^{0.85}.$$

So,

$$19.5 \doteq 5^{1.85}$$

and

$$\begin{aligned}(19.5)^{7/8} &\doteq (5^{1.85})^{7/8} \\ &= 5^{1.85 \times (7/8)} \\ &\doteq 5^{1.62} \\ &= 5^{0.62} \times 5.\end{aligned}$$

Since, from the graph,

$$5^{0.62} \doteq 2.71,$$

we know that

$$\begin{aligned}(19.5)^{7/8} &\doteq 2.71 \times 5 \\ &= 13.55\,.\end{aligned}$$

This approximation is not bad, since, correct to the nearest hundredth, $(19.5)^{7/8} = 13.45$. [Since $13.45 - 13.55 = -0.1$, the error is about -0.1. Since $-0.1/13.45 \doteq -3/400$, the relative error is about $-3/400$. The percentage error is about $-3/4\%$.] However, we can do better if we use better approximations than those we can read from the graph.

In fact, it can be shown [later, you will see how] that

$$3.9 \doteq 5^{0.8456}.$$

Using 0.8456, instead of our earlier 0.85, we find on repeating the preceding steps, that

$$(19.5)^{7/8} \doteq 5^{0.6149} \times 5 \quad [1.8456 \times \frac{7}{8} = 1.6149].$$

A more accurate way than our graphical one of estimating $5^{0.6149}$ gives

$$5^{0.6149} \doteq 2.69.$$

So, we know that

$$\begin{aligned}(19.5)^{7/8} &\doteq 2.69 \times 5\\ &= 13.45\end{aligned}$$

--our better approximations have given us the rational approximation to $(19.5)^{7/8}$ which is correct to the nearest hundredth. [The percentage error is between -0.05% and 0.05%.]

One way of obtaining the two approximations we used in the course of the computation is to have a very large-scale graph of the inverse of the exponential function with base 5. The scale would have to be very large for us to be able to read, for example, that the value of this inverse function for the argument 3.9 is 0.8456, correct to the nearest ten-thousandth. Pretty clearly, such a graph would be impractical.

Another way of obtaining the approximations we need is to have a table which lists approximations to the values of the inverse function for lots of arguments. Such a table would also be impractical if it had to list values for all the arguments we might ever need. Fortunately, this is not necessary. For one thing, as illustrated both at the beginning and the end of the computation in the example, we need consider only arguments between 1 and 5. All other positive numbers can be brought into this range by multiplying [or dividing] by an integral power of 5. This still leaves us with a lot of arguments, but, as it turns out, we can get sufficiently good approximations to the numbers we are interested in if the table contains approximations to the values of the inverse function for a relatively small number of arguments which are evenly spaced between 1 and 5. [That we can do so is a consequence of the fact that the inverse function--as well as the exponential function itself--is continuous.]

A MORE CONVENIENT CHOICE OF BASE

Such a table would be a great help in carrying out computations but, very probably, none has ever been made. To see why, let's look again at the way such a table would be used. To begin with, unless the numbers one starts with are between 1 and 5, one must replace them by numbers in this range by multiplying by the appropriate integral powers of 5. This can be a fair amount of work in itself. For example, before using the table to compute an approximation to 0.0543×2536, you would have to multiply the first factor by 25 and divide the second by 625. A similar problem arises at the end of the computation. If, for example, you have found that the number you are computing is approximately $5^{4.6832}$ then, after using the table to find an approximation to $5^{0.6832}$, you have still to multiply this by 5^4.

Now, there is an easy way to clear up the first difficulty. Although multiplying and dividing by integral powers of 5 is not too easy, multiplication and division by integral powers of 10 is a trivial matter. This is, of couse, because we use the base-10 system of numerals. Each positive number is the product of a number between 1 and 10 [or 1] by some integral power of 10. For example, $0.0543 = 5.43 \times 10^{-2}$ and $2536 = 2.536 \times 10^3$. So, to approximate 0.0543×2536, we can, instead, approximate 5.43×2.536 and multiply the result by 10. Our table will have to give us approximations for arguments between 1 and 10, rather than merely between 1 and 5. Let's assume that it does. Now, how about the second problem--multiplying by a power of 5 at the end? This is a little more difficult. We would find, for example, that

$$5.43 \times 2.536 \doteq 5^{1.6274}.$$

Since 1.6274 is beyond the values given in our table [$5.43 \times 2.536 > 10$], we must, as at the beginning, proceed indirectly. One way is to find an approximation to $5^{0.6274}$ [from the table] and multiply this by 5. This isn't too hard, but if we had, instead, had to multiply by, say, 5^4, there wouldn't have been much point in the whole procedure. However, there is another way out. As our table might tell us,

$$10 \doteq 5^{1.4306}.$$

So, since $1.6274 - 1.4306 = 0.1968$,

$$5^{1.6274} = 5^{0.1968} \times 5^{1.4306}$$
$$\doteq 5^{0.1968} \times 10.$$

Hence, a better procedure would be to approximate $5^{0.1968}$ from the table and multiply by 10.

Fortunately, there is a still better procedure. All our difficulties have arisen from the fact that, in making use of limited tables of the inverse of the exponential function with base 5, it is most convenient to have the arguments we must deal with "reduced" to the interval from 1 to 5 by multiplying and dividing by integral powers of 5--but, because we use the decimal system of notation, it is not always easy to carry out this reduction. Although we have seen how to get around this trouble, it should be evident that the most simple way out is through it. Either change to the base-5 system of numeration--in which case, multiplication and division by integral powers of 5 would be as simple as multiplication and division by integral powers of 10 is in the decimal system--or use the inverse of the exponential function with base 10. Since the decimal system of numeration is well entrenched, we shall adopt the latter alternative. This requires that we have a table of approximations to the values of the inverse of the exponential function with base 10 for closely spaced arguments between 1 and 10. There is such a table on pages 9-367 through 9-369. To see how to use it, let's recalculate $(19.5)^{7/8}$. From the table,

$$1.95 \doteq 10^{0.2900}.$$

So, since $19.5 = 1.95 \times 10^1$,

$$19.5 \doteq 10^{1.2900}$$

and

$$(19.5)^{7/8} \doteq (10^{1.2900})^{7/8}$$
$$\doteq 10^{1.1288}$$
$$= 10^{0.1288} \times 10.$$

Now, 0.1288 is not listed in the table among the approximations to the values of the inverse of the exponential function with base 10. However, from the table, we see that

$$10^{0.1271} \doteq 1.34 \quad \text{and} \quad 10^{0.1303} \doteq 1.35.$$

Since 0.1288 is about midway between 0.1271 and 0.1303, we shall not be far off in assuming that $10^{0.1288}$ is midway between 1.34 and 1.35:

$$10^{0.1288} \doteq 1.345.$$

Hence,

$$\begin{aligned}(19.5)^{7/8} &\doteq 1.345 \times 10\\ &= 13.45.\end{aligned}$$

[As previously remarked, this result is correct to the nearest hundredth.]

To check your understanding of the table, use it to verify the following statements.

$$2.00 \doteq 10^{0.3010}$$

$$2.01 \doteq 10^{0.3032}$$

$$2.02 \doteq 10^{0.3054}$$

$$4.37 \doteq 10^{0.6405}$$

$$10^{0.9390} \doteq 8.69$$

$$10^{0.0128} \doteq 1.03$$

$$10^{3.5527} \doteq 3.57 \times 10^3$$

$$85300 \doteq 10^{4.9309}$$

Use two of the approximations you have checked and another one from the table to obtain an approximation to 8.59×1.03.

EXERCISES

A. Use the table on pages 9-367 through 9-369 to find approximations to the roots of the following equations.

1. $10^{0.3979} = x$

2. $10^{1.3979} = x$

3. $10^{4.3979} = x$

4. $10^{-0.6021} = x$

[Hint for Exercise 4. $10x = 10^{1 - 0.6021}$]

5. $10^y = 848$

6. $10^y = 84800$

7. $10^y = 8.48$

8. $10^y = 0.848$

9. $10^{0.9978} = x$

10. $10^{0.7042} = x$

11. $10^{y} = 136$

12. $10^{y} = 4360$

13. $10^{y} = 110$

14. $10^{y} = 101$

15. $10^{y} = 1080$

16. $10^{y} = 1800$

<u>Sample 1.</u> $10^{y} = 9.5492$

The number 9.5492 is not listed in the table as an argument. However, 9.55 is one of the listed arguments and

$$9.5492 \doteq 9.55.$$

From the table,

$$9.55 \doteq 10^{0.9800}.$$

So, the root of the given equation is approximately 0.98.

<u>Sample 2.</u> $10^{0.7453} = x$

The number 0.7453 is not listed in the table as a value. However, 0.7451 is one of the listed values and

$$0.7453 \doteq 0.7451.$$

Hence,

$$10^{0.7453} \doteq 10^{0.7451}$$

and since, from the table,

$$10^{0.7451} \doteq 5.56,$$

the root of the given equation is approximately 5.56.

17. $10^{0.6308} = x$

18. $10^{0.7610} = x$

19. $10^{1.0056} = x$

20. $10^{2.4899} = x$

21. $10^{0.4769} = x$

22. $10^{y} = 6.3592$

23. $10^{y} = 4.3152$

24. $10^{y} = 521.62$

25. $10^{y} = 6382419$

26. $10^{y} = 1000010$

B. Use the table for the inverse of the exponential function with base 10 in carrying out the computations required in the following problems. [The volume and area formulas you need to set up some of the problems are discussed in Appendix D, pages 9-270 through 9-312.]

Sample 1. Find the area of a rectangle which is 5.64 feet wide and 18.32 feet long.

Solution. We need to find the product of 18.32 by 5.64.

$$
\begin{aligned}
18.32 \times 5.64 &\doteq 18.3 \times 5.64 \\
&= 1.83 \times 5.64 \times 10 \\
&\doteq 10^{0.2625} \times 10^{0.7513} \times 10 \\
&= 10^{(0.2625+0.7513+1)} \\
&= 10^{2.0138} \\
&= 10^{0.0138} \times 10^{2} \\
&\doteq 1.03 \times 10^{2} \\
&= 103
\end{aligned}
$$

The area is approximately 103 square feet.

Sample 2. Find the area of a circle whose radius is 6.08 inches long.

Solution. We need to compute $\pi(6.08)^2$.

$$
\begin{aligned}
\pi(6.08)^2 &\doteq 3.14 \times (6.08)^2 \\
&\doteq 10^{0.4969}(10^{0.7839})^2 \\
&= 10^{0.4969} \times 10^{1.5678} \\
&= 10^{2.0647} \\
&= 10^{0.0647} \times 10^{2} \\
&\doteq 116
\end{aligned}
$$

The area is approximately 116 square inches.

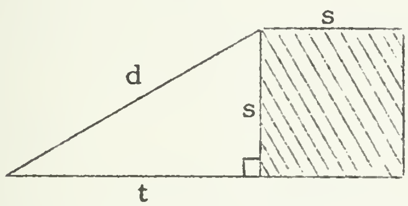
1. Compute the area of a rectangular region which is 32.6 feet long and 17.2 feet wide.

2. Find the perimeter of the rectangle of Exercise 1.

3. What is the area of a circular region whose diameter is 29.8 centimeters long?

4. Compute the circumference of the circle of Exercise 3.

5.



If $d = 38.6$ and $t = 27.4$, what is the area-measure of the square region? [Hint. Factor '$d^2 - t^2$'.]

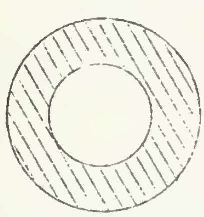
6. Find the area-measure of the region bounded by a circular sector whose central angle is an angle of 172° and whose radius is 7.36.

7. The side-measures of a triangle are 38.2, 46.3 and 68.7, respectively. Compute the area-measure of the region bounded by the triangle.

8. The hypotenuse of a right triangle is 25.3 inches long and one of the acute angles is an angle of 72°. Compute the measures of the legs and the area-measure of the region bounded by the triangle.

9. How many cubes 2 inches on a side can be stacked in a rectangular box whose inside dimensions are 8 feet 6 inches, 11 feet 4 inches, and 17 feet 10 inches?

10.



Compute the area-measure of the circular ring whose inner radius is 3.27 and whose outer radius is 5.68.

11. Solve the system of equations:

$$9.82x + 7.63y = 11.2$$
$$4.83x + 9.21y = 19.3$$

12. Compute an approximation to $118!/114!$.

LINEAR INTERPOLATION

You have seen one way in which, given an argument [of the inverse of the exponential function with base 10] which is not listed in the table, you can still use the table to get an approximation to the value of the inverse function for the given argument. Just take the closest approximation to the given argument which is listed, and use the listed approximation to the corresponding value. For most purposes one needs better approximate values than are given by this method. Linear interpolation is a method of getting much better approximations. Here's how it works.

Suppose that we want an approximation to the root of:

$$(1) \qquad 10^y = 3.443$$

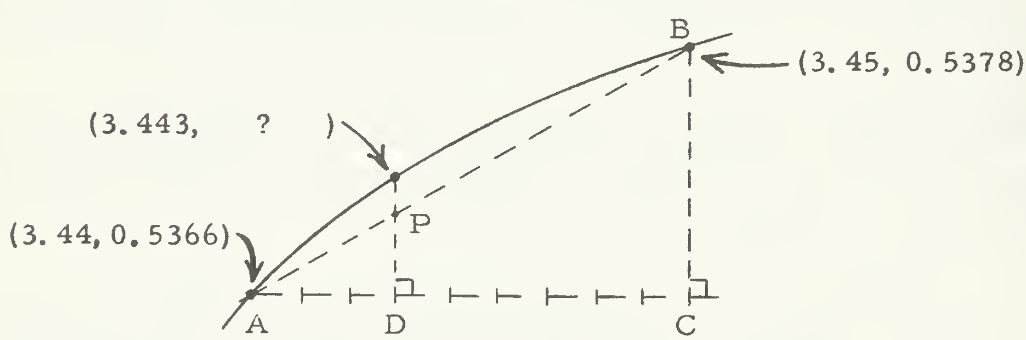
Now, 3.443 is not listed as an argument in the table, but

$$3.44 < 3.443 < 3.45$$

and, according to the table,

$$3.44 = 10^{0.5366} \qquad \text{and} \qquad 3.45 = 10^{0.5378}.$$

Drawn to a large scale, a piece of a graph of the inverse of the exponential function with base 10 looks something like this:



Up to now, we have used the ordinate of A, 0.5366, as an approximation to the root of (1). However, from the sketch, it is clear that the ordinate of the point P whose abscissa is 3.443 and which lies on the chord $\overleftrightarrow{AB}$ is a better approximation to the root of (1). Using similar right triangles--specifically, the similarity APD ⟷ ABC--it is not hard to find DP. Adding this to 0.5366, we have the ordinate of P. Here is how one works it out.

Due to the similarity, we have DP/CB = AD/AC and, so,

$$\begin{aligned} DP &= CB \cdot \frac{AD}{AC} \\ &= (0.5378 - 0.5366) \cdot \frac{3.443 - 3.44}{3.45 - 3.44} \\ &= 0.0012 \times \frac{0.003}{0.01} = 0.0012 \times 0.3 \\ &= 0.00036 \doteq 0.0004. \end{aligned}$$

Hence, the ordinate of P is approximately 0.5366 + 0.0004 and, so, 0.5370 is an approximation to the root of (1).

Here is a convenient form in which to arrange the work. [With practice, you may find that you can do it all in your head.] Notice the neglect of decimal points. [They are usually omitted, also, from tables like the one we are using.]

		3450	5378			
10		3443	5370		12	$12 \times \frac{3}{10} = 3.6 \doteq 4$
	3			4		
		3440	5366			[5366 + 4 = 5370]

Explanation: The argument we are interested in is 3 tenths of the way from the smaller to the larger of the listed arguments. So, the value we want is about 3 tenths of the way from the value for the smaller listed argument to that for the larger. The difference between the two values is 12 [really, 0.0012] and 3 tenths of this is about 4. So, add 4 to 5366--

$$10^{0.5370} \doteq 3.443.$$

The same procedure works backwards to find an approximation to the root of:

$$10^{0.5370} = x$$

		3450	5478			
10		3443	5370		12	$\frac{4}{12} = \frac{1}{3} \doteq 0.3$
	3			4		
		3440	5366			

Since the value we are interested in is about 3 tenths of the way between the listed values, the argument we want will be about 3 tenths of the way between the corresponding listed arguments.

EXERCISES

A. Use linear interpolation to approximate the missing component.

1. (6.387,   )
2. (, 0.8440)
3. (3.483,   )
4. (, 0.9935)
5. (0.8256,   )
6. (, 0.8256)
7. (58.32,   )
8. (, 1.5121)
9. (7.805,   )
10. (, -0.3723)

[Hint for Exercise 10. $10^{-0.3723} = 10^{0.6277} \times 10^{-1}$]

> Note: From now on, use linear interpolation to find approximations to arguments and values not listed in the table.

B. Use the table for the inverse of the exponential function with base 10 in carrying out the computations required in the following problems. [The volume and area formulas you need to set up some of the problems are discussed in Appendix D.]

Sample. Find the area of the triangle whose base is 0.006523 feet long and whose height is 0.05326 feet.

Solution.
$$\frac{1}{2} \times 0.006523 \times 0.05326$$
$$= (5 \times 10^{-1}) \times (6.523 \times 10^{-3}) \times (5.326 \times 10^{-2})$$
$$= 5 \times 6.523 \times 5.326 \times 10^{-6}$$
$$\doteq 10^{0.6990} \times 10^{0.8144} \times 10^{0.7264} \times 10^{-6}$$
$$= 10^{2.2398 - 6}$$
$$= 10^{0.2398 - 4}$$
$$\doteq 1.737 \times 10^{-4}$$

The area of the triangle is approximately 0.0001737 square feet.

1. Find the area of the region enclosed by a parallelogram which has a base 0.005624 feet long and a height of 0.07893 feet.

2. Find the volume of a rectangular solid whose dimensions are 7.802 meters, 0.9905 meters, and 3.01 meters.

3. Find the volume of a cone whose base has a radius of 9.86 meters and whose height is 17.2 centimeters.

4. Find the volume of a solid sphere whose radius is 0.9124 feet long.

5. Find the surface area of the sphere in Exercise 4.

6. Find the volume of a cylinder whose base has a radius of 0.2745 feet and whose height is 0.8183 feet.

7. Find the lateral area of the cylinder of Exercise 6.

8. Find the total area of the cylinder of Exercise 6.

9. Find the area of a trapezoidal region whose bases are 3.768 and 0.5892 centimeters long and whose height is 60.14 centimeters.

10. Find the volume of a cube of edge-length 0.5185 centimeters.

11. Find the area of the lateral surface of a right circular cone whose radius is 1.247 feet and whose slant height is 4.687 feet.

12. Find the total area of the cone of Exercise 11.

13. Find the weight of an iron washer whose outer diameter, inner diameter, and thickness are 13.5 inches, 8.74 inches, and 0.215 inches, respectively. [Iron weighs about 488.8 lbs./cubic foot.]

14. Find the total surface of a solid hemisphere whose diameter is 6.875 inches.

EXPLORATION EXERCISES

A. Let's use '$\exp_5$' to name the exponential function with base 5. Then we can use functional notation. For example:

$$\exp_5(2) = 5^2 = 25 \quad \text{and} \quad \exp_5(-3) = \frac{1}{125}$$

Complete.

1. $\exp_5(3) =$ 2. $\exp_5(1) =$ 3. $\exp_5(\quad) = 0.2$

4. $\exp_5(\quad) = 0.04$ 5. $\exp_5(\frac{1}{2}) =$ 6. $\exp_5(\quad) = \frac{\sqrt{5}}{5}$

7. $\exp_5(1.5) =$ 8. $\exp_5(\pi) =$ 9. $\exp_5(\quad) = 1$

10. $\exp_5 = \{(x, y): y = \qquad\}$

11. $\exp_5(\quad) = 0$ 12. $\exp_5(-2) =$ 13. $\exp_5(\quad) = -2$

14. $\mathcal{D}_{\exp_5} = \{x: \qquad\}$ 15. $\mathcal{R}_{\exp_5} = \{y: \qquad\}$

16. $\exp_5(3) \cdot \exp_5(4) = \exp_5(\quad)$ 17. $\exp_5(18) = \exp_5(12) \cdot \exp_5(\quad)$

18. $\exp_5(12) = \exp_{25}(\quad)$ 19. $25 \cdot \exp_5(\pi) = \exp_5(2 + \quad)$

20. Suppose that $\exp_5(a) = b$. It follows that $b \cdot \exp_5(\pi) = \exp_5(\qquad)$.

B. By Theorems 202 and 184, $\exp_5$ is monotonic and has an inverse.

Sample. [the inverse of $\exp_5$](125) = ?

Solution. We are looking for the number a such that $\exp_5(a) = 125$. Since $\exp_5(3) = 125$, it follows that [the inverse of $\exp_5$](125) = 3.

1. [the inverse of $\exp_5$](25) = ? 2. [the inverse of $\exp_5$](5) = ?

3. [the inverse of $\exp_5$](0) = ? 4. [the inverse of $\exp_5$](?) = −1

*

The inverse of $\exp_5$ is usually called <u>the logarithm function to the base 5</u>, and this is abbreviated to '$\log_5$'. Thus, $\log_5(125) = 3$. [Read this as 'the logarithm to the base 5 of 125 is 3', or as 'the logarithm of 125 to the base 5 is 3', or as 'log sub-five of 125 is 3'.]

*

5. $\log_5(625) = ?$ 6. $\log_5(0) = ?$ 7. $\log_5(0.04) = ?$

8. $\log_5(\ ?\ ) = \frac{1}{2}$ 9. $\log_5(5^6) = ?$ 10. $\log_5(\ ?\ ) = 1.5$

11. $\log_5(\exp_5(2)) = ?$ 12. $\exp_5(\log_5(\ ?\ )) = 4$

13. $[\log_5 \circ \exp_5](17) = ?$ 14. $[\exp_5 \circ \log_5](\ ?\ ) = ?$

15. $[\log_5 \circ \exp_5](-9) = ?$ 16. $[\exp_5 \circ \log_5](-9) = ?$

17. $\log_5(\ ?\ ) = -1$ 18. $\log_5(0) = ?$ 19. $\log_5(-1) = ?$

20. The domain of $\log_5$ is $\{x: \qquad\qquad \}$.

21. The range of $\log_5$ is $\{y: \qquad\qquad \}$.

22. $\log_5 = \{(x, y): \qquad\qquad \}$

23. $\exp_5(\log_5(7)) = ?$ 24. $5^{\log_5(7)} = ?$

25. $5^{\log_5(24)} = ?$ 26. $5^{\log_5(6)} = ?$ 27. $5^{\log_5(4)} = ?$

28. $5^{(\log_5(6) + \log_5(4))} = ?$

29. $\log_5(6) + \log_5(4) = \log_5(\ ?\ )$

30. $\log_5(48) = \log_5(12) + \log_5(\ ?\ ) = \log_5(16) + \log_5(\ ?\ )$

31. $\log_5(22) = \log_5(11) + \log_5(\ ?\ ) = \log_5(44) - \log_5(\ ?\ )$

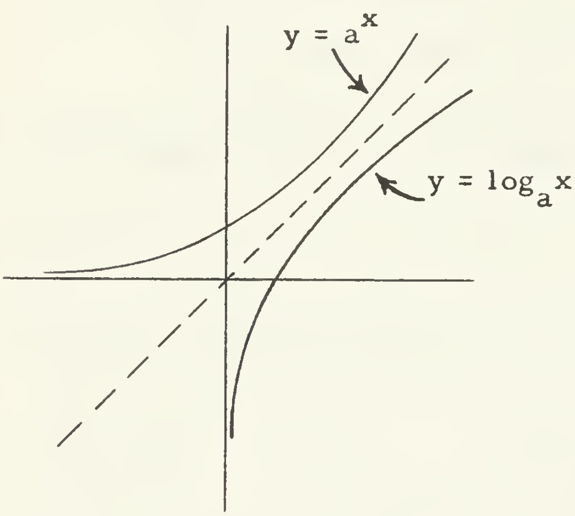
32. $\log_5(22) = \log_5(66) + \log_5(\ ?\ )$

9.09 The logarithm functions.--In the preceding section you learned how to use the inverse of an exponential function to translate problems involving multiplication, division, or exponentiation into problems which require only addition, subtraction, or multiplication, respectively. These "inverse exponential functions" are called logarithm functions. Specifically, for $0 < a \neq 1$, the inverse of the exponential function with base a is the logarithm function to the base a. Since, for $a > 0$, the exponential function with base a is

$$\{(x, y): y = a^x\},$$

it follows that, for $0 < a \neq 1$, the logarithm function to the base a is

$$\{(x, y): a^y = x\}.$$



These graphs show an exponential function with base $a > 1$ and the logarithm function to the same base. Make a sketch to illustrate the case in which $0 < a \neq 1$. Also, sketch the exponential function with base 1 and its converse.

That exponential functions have inverses--that is, that there are logarithm functions--follows from Theorem 202 and Theorem 184. [Why isn't there a logarithm function to the base 1? To a negative base?]

Here are pairs of equivalent statements [fill in the missing statement]:

4 = the logarithm to the base 2 of 16 .... $2^4 = 16$

1 = the logarithm to the base 7 of 7 .... $7^1 = 7$

3 = the logarithm to the base 4 of 64 .... ________

$0.3010 \doteq$ the logarithm to the base 10 of 2 ... $10^{0.3010} \doteq 2$

0 = the logarithm to the base π of 1 .... $\pi^0 = 1$

-4 = the logarithm to the base 1/2 of 16 ... $(1/2)^{-4} = 16$

[The statements in the first column are customarily abbreviated to:

$$4 = \log_2 16, \quad 1 = \log_7 7, \quad 3 = \log_4 64, \quad 0.3010 \doteq \log 2, \quad \ldots$$

Note the fourth of these abbreviated statements. '$\log_{10}$' is usually abbreviated to 'log', and the logarithm function to the base 10 is often called <u>the common logarithm function</u>. Tables like the one you have been using on pages 9-367 and 9-369 are called <u>tables of common logarithms</u>.]

As illustrated, for $0 < a \neq 1$ and $b > 0$, the number $\log_a b$ is

the number z such that $a^z = b$.

In order to prove theorems about the logarithm functions we adopt the defining principle:

$$\text{(L)} \qquad \forall_{0 < a \neq 1} \forall_{x > 0}\ a^{\log_a x} = x$$

As for any inverse function, we have a uniqueness theorem. Here is a proof:

Since, by Theorems 202 and 184, the exponential function with base a has [for $0 < a \neq 1$] an inverse, we know that

$$\forall_{0 < a \neq 1} \forall_y \forall_z\ [a^y = a^z \Rightarrow y = z].$$

So [taking '$\log_a x$' for 'z' and using (L)], we obtain the uniqueness theorem:

$$\forall_{0 < a \neq 1} \forall_{x > 0} \forall_y\ [a^y = x \Rightarrow y = \log_a x] \qquad \text{[Theorem 212]}$$

Notice that a more complicated way of stating what (L) says is:

$$\forall_{0 < a \neq 1} \forall_{x > 0} \forall_y\ [y = \log_a x \Rightarrow a^y = x]$$

So, we can think of Theorem 212 as the converse of (L). Together they tell us that, for example, the sentences:

$$5^3 = 125 \qquad \text{and:} \qquad 3 = \log_5 125$$

are equivalent. So, a sentence of the form of either of these can be translated into a sentence of the form of the other.

EXERCISES

A. Here are sentences of the forms illustrated above. Translate each into a sentence of the other form.

1. $6^3 = 216$
2. $3^6 = 729$
3. $10^1 = 10$
4. $\log_{10} 100 = 2$
5. $\log 1000 = 3$
6. $\log_5 3.5 \doteq 0.78$
7. $10^{0.477} \doteq 3$
8. $2^{10} = 1024$
9. $10^{0.0781} \doteq 1.197$
10. $\log_x 17 = 2.3$
11. $\log_5 y = 13$
12. $\log 1.19 \doteq 0.0755$
13. $10^a = 50$
14. $3^a = b$
15. $a^b = c$
16. $\log_4 5 = a$
17. $\log_3 b = a$
18. $\log_a c = b$
19. $5^0 = 1$
20. $0^8 = 0$
21. $1^6 = 1$
22. $\log_9 3 = \frac{1}{2}$
23. $\log_3 \frac{1}{9} = -2$
24. $\log_{1/5} 3.89 \doteq -0.85$
25. $\sqrt[4]{81} = 3$
26. $81 = 3^4$
27. $4^2 = 2^4$
28. $27^{\frac{2}{3}} = 9$
29. $64^{\frac{5}{6}} = 32$
30. $64^{-\frac{5}{6}} = \frac{1}{32}$

B. Simplify.

1. $\pi^{\log_\pi 4}$
2. $10^{\log 4}$
3. $0.5^{\log_{0.5} 4}$
4. $2^{(\log_2 3 + \log_2 5)}$
5. $3^{(\log_3 7 - \log_3 2)}$
6. $\left(6^{\log_6 \pi}\right)^3$
7. $6^{3 \log_6 \pi}$
8. $9^{5 \log_9 2}$
9. $4^{\log_2 5}$
10. $5^{-\log_5 7}$
11. $5^{\log_5 \frac{1}{7}}$

C. Use Theorem 212 to complete each of the following.

1. Since $5^0 = 1$,
   it follows that $0 = \log_5$ ____.

2. Since $\qquad \pi^1 = \pi,$

it follows that $\qquad \_\_\_ = \log_\pi \_\_\_.$

3. Since $\qquad 2^{(\log_2 3 + \log_2 5)} = 3 \cdot 5,$

it follows that $\qquad \log_2 3 + \log_2 5 = \_\_\_\_\_.$

4. Since $\qquad 3^{(\log_3 7 - \log_3 2)} = \frac{7}{2},$

it follows that $\qquad \log_3 7 - \log_3 2 = \_\_\_.$

5. Since $\qquad 6^{3\log_6 \pi} = \pi^3,$

it follows that $\qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$

6. Since $\qquad 5^{-\log_5 7} = \frac{1}{7},$

it follows that $\qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$

THEOREMS ABOUT LOGARITHM FUNCTIONS

Using (L) and Theorems 184, 187, and 182 one can prove:

> <u>Theorem 213.</u>
>
> The domain of each logarithm function is the set of positive numbers and its range is the set of real numbers. Each such function is continuous and monotonic--decreasing if its base is between 0 and 1 and increasing if its base is greater than 1.

Since, for any a, $a^0 = 1$ and $a^1 = a$, it follows from Theorem 212 that

$$\forall_{0 < a \neq 1} (\log_a 1 = 0 \text{ and } \log_a a = 1) \qquad \text{[Theorem 214].}$$

In a similar way [the same way in which the theorems about the principal root functions were proved], you can prove theorems which justify the computational uses you made of logarithms. For example:

$$\forall_{0 < a \neq 1} \forall_{x > 0} \forall_{y > 0} \log_a(xy) = \log_a x + \log_a y \qquad \text{[Theorem 215]}$$

Proof. For $b > 0$ and $c > 0$, by Theorem 205 [on the exponential functions],

$$a^{(\log_a b + \log_a c)} = a^{\log_a b} \cdot a^{\log_a c}.$$

Since $b > 0$ and $c > 0$, it follows, by (L), that

$$a^{\log_a b} \cdot a^{\log_a c} = b \cdot c.$$

Consequently,

$$a^{(\log_a b + \log_a c)} = bc.$$

Since $b > 0$ and $c > 0$, $bc > 0$ and, so, by Theorem 212,

$$\log_a (bc) = \log_a b + \log_a c.$$

EXERCISES

A. Prove.

1. $\forall_{0 < a \neq 1} \forall_{x > 0} \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ [Theorem 216]

2. $\forall_{0 < a \neq 1} \forall_{x > 0} \forall_u \log_a (x^u) = u \cdot \log_a x$ [Theorem 217]

3. $\forall_{0 < a \neq 1} \forall_{x > 0} \log_a (1/x) = -\log_a x$ [Theorem 218a]

☆4. $\forall_{0 < a \neq 1} \forall_{0 < b \neq 1} \log_b a = 1/\log_a b$ [Theorem 218b]

[Hint. In Theorem 217, take 'b' for 'x' and '$\log_b a$' for 'u'.]

☆5. If c is a sequence with positive terms then, for each n,

$$\log_a \left(\prod_{p=1}^{n} c_p \right) = \sum_{p=1}^{n} \log_a c_p.$$

B. 1. Use the facts that $\log 2 \doteq 0.30103$ and $\log 3 \doteq 0.47712$ to make a table of logarithms of the integers from 1 to 9. [Hint. The only entry difficult to compute is log 7. One way to approximate this is to use the fact that $48 < 49 < 50$. You may be able to find a better way.]

2. Solve for 'x': $\log_b x \cdot \log_5 b = 3$

3. Show that, for $0 < a \neq 1$ and $x > 1$,

$$\log_a\left(x - \sqrt{x^2 - 1}\right) = -\log_a\left(x + \sqrt{x^2 - 1}\right).$$

4. Show that, for $x > 1$ and for $x < 0$,

$$\log\left(1 - \frac{1}{x}\right) = -\log\left(1 + \frac{1}{x - 1}\right).$$

C. Use the table of common logarithms on pages 9-367 through 9-369 to find an approximation to the solution of each of the following equations.

Sample 1. $\log 672.1 = x$

Solution. Since $672.1 = 6.721 \times 10^2$,

$$\begin{aligned}\log 672.1 &= \log 6.721 + \log 10^2 \\ &= \log 6.721 + 2.\end{aligned}$$

From the table, using linear interpolation, $\log 6.721 \doteq 0.8275$. So, the root of the equation is approximately 2.8275.

Sample 2. $\log x = -2.7$

Solution. $\log x = -2.7$ if and only if

$\log(x \cdot 10^3) = -2.7 + 3 = 0.3$.

From the table we see that this is the case if and only if $x \cdot 10^3 \doteq 1.995$. So, the root is approximately 0.001995.

1. $\log 3 = x$
2. $\log 3.455 = x$
3. $\log 34.55 = x$
4. $\log 345500 = x$
5. $\log 0.003455 = x$
6. $\log 0.0618 = x$
7. $\log x = 2.3990$
8. $\log x = 0.8386$
9. $\log x = -1.4425$
10. $\log x = 3.6618 - 5$
11. $\log 0.000001067 = x$
12. $\log 0.5959 = x$
13. $\log 1000 = x$
14. $\log x = 0$

SCIENTIFIC NOTATION

As you have seen while using common logarithms for computation, you can find log u, for any $u > 0$, if you know the values of log for arguments from 1 up to 10. Just express the number in scientific notation and use Theorem 215.

For example, suppose that u is 984.7. Then,

$$\log u = \log 984.7 = \log(9.847 \times 10^2) = \log \underbrace{9.847} + \log 10^2.$$

Similarly, (between 1 and 10)

$$\log 0.0004356 = \log(4.356 \times 10^{-4}) = \log \overbrace{4.356} + \log 10^{-4}.$$

Now, an interesting question to consider is whether each positive number can be expressed in scientific notation--that is, given a number $u > 0$, is it possible to find a number v such that $1 \leq v < 10$ and an integer k such that

$$u = v \times 10^k?$$

As you might suspect, the answer is 'yes' and we can use our knowledge of the greatest-integer function and of the common logarithm function to prove it.

Recall that, for each x,

$$x = \llbracket x \rrbracket + \{\!\{x\}\!\},$$

where $\llbracket x \rrbracket$, the integral part of x, is the integer k such that $k \leq x < k + 1$, and $\{\!\{x\}\!\}$, the fractional part of x, is $x - \llbracket x \rrbracket$, and, so, is nonnegative and less than 1. Examples:

$$46.38 = \llbracket 46.38 \rrbracket + \{\!\{46.38\}\!\} = 46 + 0.38$$

$$\frac{25}{3} = \left\llbracket \frac{25}{3} \right\rrbracket + \left\{\!\!\left\{\frac{25}{3}\right\}\!\!\right\} = 8 + \frac{1}{3}$$

$$9\pi = \llbracket 9\pi \rrbracket + \{\!\{9\pi\}\!\} = 28 + 0.2744^-$$

$$-1.765 = \llbracket -1.765 \rrbracket + \{\!\{-1.765\}\!\} = -2 + 0.235$$

Now, suppose that we are given a number $u > 0$. Since u is a positive number, it has a common logarithm. And, of course, this common logarithm has an integral part and a fractional part.

$$(1) \qquad \log u = \llbracket \log u \rrbracket + \{\!\{\log u\}\!\},$$

where $[\![\log u]\!]$ is the integer k such that $k \leq \log u < k + 1$ and, consequently, $0 \leq \{\!\{\log u\}\!\} < 1$. Now, by (L),

$$u = 10^{\log u}. \qquad (2)$$

So, from (1) and (2),

$$\begin{aligned} u &= 10^{([\![\log u]\!] + \{\!\{\log u\}\!\})} \\ &= 10^{\{\!\{\log u\}\!\}} \times 10^{[\![\log u]\!]}. \end{aligned}$$

Since $0 \leq \{\!\{\log u\}\!\} < 1$, and since log is increasing,

$$10^0 \leq 10^{\{\!\{\log u\}\!\}} < 10^1. \qquad (3)$$

Hence,

$$u = v \times 10^k,$$

where $v = 10^{\{\!\{\log u\}\!\}}$ and $k = [\![\log u]\!]$. By (3), $1 \leq v < 10$, and, since k is the integral part of a number, k is an integer.

CHARACTERISTICS AND MANTISSAS

It is customary to refer to the integral part of log u [for $u > 0$] as the <u>characteristic</u> of log u and to refer to the fractional part of log u as its <u>mantissa</u>.

What we have shown in our discussion of scientific notation is that, given $u > 0$,

the characteristic of log u is the greatest integer k such that $10^k \leq u$

and

the mantissa of log u is, for this k,

$$\log\left(\frac{u}{10^k}\right).$$

The simplest shortcut for finding the characteristic and mantissa of the common logarithm of a number u is to express the number in scientific notation:

$$u = v \times 10^k$$

characteristic of log u

mantissa of log u is log v

For example, since

$$243.1 = 2.431 \times 10^2,$$

the characteristic of log 243.1 is 2 and the mantissa of log 243.1 is log 2.431. Also, since

$$0.002431 = 2.431 \times 10^{-3},$$

the characteristic of log 0.002431 is −3 and the mantissa of log 0.002431 is, again, log 2.431. So,

$$\log 243.1 = 2 + \log 2.431 \doteq 2 + 0.3858 = 2.3858$$

and

$$\log 0.002431 = -3 + \log 2.431 \doteq -3 + 0.3858 = -2.6142.$$

<u>A word of warning about negative characteristics</u>: Since, for example,

$$\log 3.725 \doteq 0.5711,$$

it follows that

$$\log 372.5 \doteq 2 + 0.5711$$

and

$$\log 0.3725 \doteq -1 + 0.5711.$$

The warning is to note that, while

$$2 + 0.5711 = 2.5711,$$

$$-1 + 0.5711 \neq -1.5711. \qquad \text{[Explain.]}$$

In fact, in standard decimal notation,

$$\log 0.3725 \doteq -0.4289.$$

Since it is usually desirable to keep track of the mantissas of one's logarithms, it is usually better <u>not</u> to use the standard decimal notation in the case of a negative characteristic. What does often turn out to be convenient is to replace '−1', say, by some such equivalent numeral as '9 - 10', '2 - 3', or '19 - 20', and simplify. The suggested replacements lead to:

$$\begin{aligned}\log 0.3725 &\doteq 9.5711 - 10\\ &= 2.5711 - 3\\ &= 19.5711 - 20\end{aligned}$$

[Some texts abbreviate '−1 + 0.5711' to '$\overline{1}$.5711', putting the oppositing sign over the '1'. Our advice to you is: Don't.]

EXERCISES

A. Use the tables of common logarithms to find approximations.

1. $\log 5.42$
2. $\log 5420$
3. $\log 0.000542$
4. $\log (5.42 \times 10^{-4})$
5. $\log 32.56$
6. $\log 0.3256$
7. $\log 0.007652$
8. $\log 763.5$
9. $10^{0.4771}$
10. $10^{2.4771}$
11. $10^{-2+0.4771}$
12. $10^{-1.5229}$

Note: The exponential function with base 10 is sometimes called the antilogarithm function. So, for example, $\text{antilog}\, 3.6921 = 10^{3.6921}$.

13. antilog 1.5569
14. antilog (9.5569 - 10)
15. antilog 2.4807
16. antilog 0.0837
17. antilog (7.8125 - 10)
18. antilog 4.9409
19. antilog (2.5308 - 3)
20. antilog (9.0000 - 10)

B. 1. What can you say about a number if the characteristic of its common logarithm is negative? Zero? Positive?

2. Suppose that g is the function such that, for each $x > 0$, $g(x)$ is the characteristic of $\log x$.

   (a) What is $g(19)$? $g(29)$? $g(309)$? $g(3090)$? $g(0.002)$?

   (b) Complete: $\forall_k \forall_{x>0}\; g(10^k x) = \qquad +$

   (c) Does g have an inverse?

   (d) What are the domain and range of g?

   (e) Find b given that $g(a) = \log b$ and $10 \le a < 100$.

3. A digital computer computed powers of 2 and kept track of the number of digits in their decimal representations. It turned out that 2^{1000} has 302 digits. Use this to find the decimal representation of log 2 correct to 3 decimal places.

4. How many digits are in the decimal representation of 3^{10000}?

C. Use common logarithms to carry out the indicated computations.

Sample 1. 0.52×13.7

Solution. $\log (0.52 \times 13.7) = \log 0.52 + \log 13.7$

$$\begin{aligned} \log 0.52 &\doteq 9.7160 - 10 \\ (+)\ \log 13.7 &\doteq \underline{1.1367} \\ \log (0.52 \times 13.7) &\doteq 10.8527 - 10 \\ 0.52 \times 13.7 &\doteq 7.123 \end{aligned}$$

[You will save time if, for each exercise, you begin by making out a form including, as above, everything not written in handscript. Then, turn to the tables and fill in the blanks.]

1. 35.5×1.28
2. 0.0235×3.79
3. 1.732×463
4. 52.74×0.00034
5. $(\log 27)(\log 3)$
6. -265×32.7
7. 2.083×-0.0931
8. -5.31×-6.82

Sample 2. $23.1 \div 0.762$

Solution. $\log (23.1 \div 0.762) = \log 23.1 - \log 0.762$

$$\begin{aligned} \log 23.1 &\doteq 1. \\ (-)\ \log 0.762 &\doteq \underline{9.\qquad - 10} \\ \log (23.1 \div 0.762) &\doteq \\ 23.1 \div 0.762 &\doteq \end{aligned}$$

[Fill in the form. When you come to subtract, note that you can make things easier by recording the characteristic of log 23.1 as '11.   - 10' in place of the '1'.]

9. $29.2 \div 17$
10. $4.63 \div 64$
11. $76.53 \div 1.942$
12. $-378.1 \div 76.5$
13. $\log 125 \div \log 47$
14. $6432 \div 9657$
15. $0.09217 \div 3.821$
16. $-4.91 \div -2.83$

Sample 3. $\dfrac{327 \times 4.8}{23}$

Solution. $\log \dfrac{327 \times 4.8}{23} = \log 327 + \log 4.8 - \log 23$

$$\begin{aligned}
\log 327 &\doteq 2.5145 \\
(+)\ \log 4.8 &\doteq \underline{0.6812} \\
\log 327 + \log 4.8 &\doteq 3.1957 \\
(-)\ \log 23 &\doteq \underline{1.3617} \\
\log \frac{327 \times 4.8}{23} &\doteq 1.8340 \\
\frac{327 \times 4.8}{23} &\doteq 68.33
\end{aligned}$$

[With a little practice you can learn to omit the line '$\log 327 + \log 4.8 \doteq$ ' from your form and do the adding and subtracting in one step.]

17. $\dfrac{9.235 \times 16.42}{563}$

18. $\dfrac{0.873 \times 5.421}{3.142}$

19. $\dfrac{35.72}{17.3 \times 0.834}$

Sample 4. $(0.563)^2$

Solution. $\log (0.563)^2 = 2 \log 0.563$

$$\begin{aligned}
\log 0.563 &\doteq 9.7505 - 10 \\
& \qquad\qquad \underline{(\times)\ 2} \\
\log (0.563)^2 &\doteq 19.5010 - 20 \\
(0.563)^2 &\doteq 0.3169
\end{aligned}$$

Sample 5. $(0.0765)^{3/4}$

Solution. $\log (0.0765)^{3/4} = \frac{3}{4} \log 0.0765$

$$\begin{aligned}
\log 0.0765 &\doteq 2.8831 - 4 \\
& \qquad\qquad \underline{(\times)\ 3} \\
(\div)\ 4 &\ \big)\ 8.6511 - 12 \\
\frac{3}{4} \log 0.0765 &\doteq 2.1628 - 3 \\
(0.0765)^{3/4} &\doteq 0.1455
\end{aligned}$$

[Note the trick of recording the characteristic, -2, of $\log 0.0765$ as '2.   - 4'; $\frac{3}{4} \times 4$ is an integer.]

Sample 6. $(3.576)^{-4}$

Solution. $\log (3.576)^{-4} = -4 \log 3.576$

$$\log 3.576 \doteq 0.5534$$

$$\underline{(\times)\ -4}$$

$$\log (3.576)^{-4} \doteq -2.2136$$

$$= 0.7864-3$$

$$(3.576)^{-4} \doteq 0.006115$$

[Note that $-2.2136 = (3 - 2.2136) - 3$.]

20. $(0.01425)^3$ 21. $(0.01425)^{1/3}$ 22. $(3.142)^2$

23. $(3.142)^{1/2}$ 24. $1/3.142$ 25. $\sqrt[5]{3.142}$

26. $\sqrt[5]{0.3142}$ 27. $\sqrt[3]{0.3142}$ 28. $\sqrt[7]{1758} \times \sqrt[6]{1707}$

29. $\sqrt[4]{0.09705} \div \sqrt[5]{0.006908}$ 30. $(1758)^{-3}$

31. $\sqrt[7]{-362}$ 32. $\sqrt[6]{-362}$ 33. $(0.0259)^{-2}$

Sample 7. $\dfrac{27^3 \times \sqrt{0.362}}{0.0259}$

Solution. $\log \dfrac{27^3 \times \sqrt{0.362}}{0.0259} = \log (27^3 \times \sqrt{0.362}) - \log 0.0259$

$$= \log 27^3 + \log \sqrt{0.362} - \log 0.0259$$

$$= 3 \log 27 + \frac{1}{2} \log 0.362 - \log 0.0259$$

$$3 \log 27 \doteq 4.2942$$

$$(+)\ \frac{1}{2} \log 0.362 \doteq 1.7794-2$$

$$(-)\ \log 0.0259 \doteq \underline{0.4133 - 2}$$

$$\log \frac{27^3 \times \sqrt{0.362}}{0.0259} \doteq 5.6603$$

$$\frac{27^3 \times \sqrt{0.362}}{0.0259} \doteq 457400$$

$$\log 27 \doteq 1.4314$$

$$\underline{(\times)\ 3}$$

$$3 \log 27 \doteq 4.2942$$

$$\log 0.362 \doteq 1.5587 - 2$$

$$\underline{(\times)\ 1/2}$$

$$\frac{1}{2} \log 0.362 \doteq 0.7794 - 1$$

$$= 1.7794-2$$

34. $\sqrt{27.9 \times 265}$

35. $\dfrac{0.156 \times 3.62^3}{\sqrt[5]{918}}$

36. $\dfrac{[46.5 \times \sqrt[3]{3.249}]^2}{734 \times 26.3}$

37. $\dfrac{(-265)^3 \times 27.9}{\sqrt[3]{-33}}$

D. Solve the following problems. [Use common logarithms if they make computations easier. See Appendix D for measurement formulas.]

Sample. Find the length of a diameter of a cast iron ball which weighs 17 pounds if a cubic foot of cast iron iron weighs about 468 pounds.

Solution. Suppose that the length of a diameter is d feet. Then, the volume of the ball is $\frac{4}{3}\pi\left(\frac{d}{2}\right)^3$ cubic feet and its weight is $\frac{4}{3}\pi\left(\frac{d}{2}\right)^3 \times 468$ pounds. So,

$$\frac{4}{3}\pi\left(\frac{d}{2}\right)^3 \times 468 = 17.$$

More simply, $\pi d^3 \times 78 = 17$. So,

$$\log \pi + 3 \log d + \log 78 = \log 17.$$

Hence,

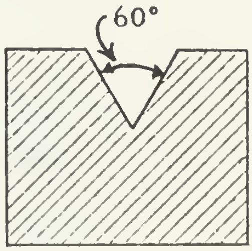
$$\log d = \frac{1}{3}[\log 17 - \log \pi - \log 78].$$

$$\begin{array}{rl}
\log 17 \doteq & 1.2304 \\
(-)\ \log \pi \doteq & \underline{0.4971} \\
 & 10.7333 - 10 \\
(-)\ \log 78 \doteq & \underline{1.8921} \\
3 \,| & 28.8412 - 30 \\
\log d \doteq & 9.6137 - 10 \qquad d \doteq 0.4109
\end{array}$$

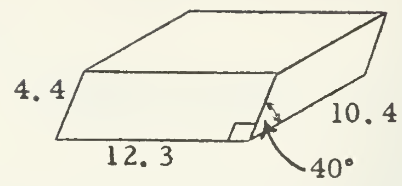
So, a diameter is about 0.41 feet in length.

1. What is the length of a radius of a spherical solid whose volume is 426 cubic inches?

2. A conical hole is to be drilled in the top of a cubical block to reduce its weight by 10%. If the apex angle of the conical hole is an angle of 60°, and the block is 25 cm. on each edge, how deep should the hole be drilled?



3. The bases of a parallelepiped are rectangular regions whose dimensions are 12.3 inches and 10.4 inches. A lateral edge of the parallelepiped is 4.4 inches long. An angle of one of the lateral faces is a right angle, and an angle of another face is an angle of 40°. Compute the volume of the parallelepiped.



4. A spherical iron shell whose outside and inside diameters are 14.3 inches and 7.2 inches, respectively, is melted and then molded into a 10.3-inch high circular cylinder. Find the length of a diameter of the cylinder.

5. The length of a diameter of a solid sphere is 18.4 inches. A cylindrical hole is bored through the center of the solid sphere in such a way that the distance between the circular openings in the surface is 16.8 inches. What is the volume of the smallest circular cylinder which could be used to fill the hole?

6. Find the total surface area and the volume of a regular tetrahedron each of whose edges is 105.7 inches long.

7. Suppose that A is the vertex of a right circular cone whose base has center O, and that B is on the boundary of the base. If $AB = 5.42$ and $m(\angle ABO) = 43$, what are the volume-measure of the cone and the area-measure of the total surface?

8. Find the net price on an article which is listed at $5.82 but is discounted successively at the rates of 12.5%, 5%, and 2.5%.

9. A man deposits $75.00 in a bank which gives interest at the rate of 4.5% compounded annually. What does the $75.00 amount to at the end of 5 years?

E. Sketch graphs of the following equations.

1. $y = \log x$ 2. $y = \log (x - 4)$ 3. $y = 2 \log x$

4. $y = \log (x + 1)^2$ 5. $y = \log x + 2$ 6. $y = \log (2x)$

7. $y = \log \frac{1}{x}$ 8 $y = -\log x$ 9. $y = \log (-x)$

10. $y = \log |x|$ 11. $y = \ \log x|$ 12. $y = (\log x)^2$

F. Each of the equations in Part E defines a function.

1. Which of these functions have inverses?

2. For each of the functions which have inverses, write a defining equation of the form '$y = \ldots$' for its inverse.

[Sample. For the function defined by '$y = \log x$', the inverse is defined by '$y = 10^x$'.]

3. (a) Solve simultaneously the equations given in Exercises 3 and 6 of Part E.

(b) Repeat for Exercises 5 and 12.

☆G. 1. Suppose that the sequence a is a geometric progression of positive terms. Show that log a is an arithmetic progression.

2. Can each arithmetic progression be obtained as in Exercise 1 from some geometric progression with positive terms?

H. Find an approximation to the root of each equation.

Sample 1. $7^u = 116$

Solution.

$$
\begin{aligned}
7^u &= 116 \\
\log 7^u &= \log 116 \\
u \log 7 &= \log 116 \\
u &= \frac{\log 116}{\log 7} \\
&\doteq \frac{2.0645}{0.8451} \\
&\doteq 2.442
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\log 2.0645 &\doteq 10.3148 - 10 \\
(-)\ \log 0.8451 &\doteq \underline{\ 9.9270 - 10} \\
\log \frac{2.0645}{0.8451} &\doteq 0.3878 \\
\frac{2.0645}{0.8451} &\doteq 2.442
\end{aligned}
}
$$

<u>Sample 2.</u> $4^x = 5^{x+1}$

<u>Solution.</u> $4^x = 5^{x+1}$

$$x \log 4 = (x+1) \log 5$$

$$x(\log 5 - \log 4) = -\log 5$$

$$x = -\frac{\log 5}{\log 5 - \log 4}$$

$$= -\frac{\log 5}{\log 1.25}$$

$$\doteq -\frac{0.6990}{0.0969}$$

$$\doteq -7.215$$

$$\boxed{\begin{aligned} \log 0.6990 &\doteq 9.8445 - 10 \\ (-)\log 0.0969 &\doteq \underline{8.9863 - 10} \\ \log \frac{0.6990}{0.0969} &\doteq 0.8582 \\ \frac{0.6990}{0.0969} &\doteq 7.215 \end{aligned}}$$

1. $3^x = 15$
2. $4^{x+2} = 265$
3. $15^{3x-4} = 7$
4. $\log_9 27 = x$
5. $5^x \cdot 3^{2x+1} = 8$
6. $5 \cdot 3^x = 4 \cdot 2^x$
7. $(5^x - 4)(5^x - 2) = 0$
8. $7^{2x} - 15 = 2 \cdot 7^x$
9. $3^x - 6 \cdot 3^{-x} = 1$
10. $10^x - 10^{-x} = 2$
11. $2^x = 3^x$

☆12. $4^x - 12 \cdot 6^{x-1} + 9^x = 0$

<u>I.</u> 1. Notice that because $3^4 = 81$, $\left(3^2\right)^{4/2} = 81$--in other words, because $\log_3 81 = 4$, $\log_{3^2} 81 = 4/2$. Complete:

(a) $\log_{3^3} 81 =$ (b) $\log_{\sqrt{3}} 81 =$ (c) $\log_{3^\pi} 81 =$

2. Use a table of common logarithms to verify each of the following.

(a) $\log_{100} 2 \doteq 0.1505$ (b) $\log_{\sqrt{10}} 2 \doteq 0.6020$

(c) $\log_{10\sqrt{2}} 34 \doteq 1.0831$ (d) $\log_{1000} 243 \doteq 0.7952$

(e) $\log_{0.1} 378 \doteq -2.5575$ (f) $\log_4 7.5 \doteq 1.2873$

[<u>Hint for part (f).</u> For what x is $4 = 10^x$?]

RELATING LOGARITHMS TO TWO BASES

The exercises of Part I suggest how to find logarithms to any base when you know logarithms to another base. For example, in doing part (f) of Exercise 2 you probably used the fact that

$$\log_4 7.5 = \frac{\log 7.5}{\log 4}.$$

As is clear from the preamble to Exercise 1, this is another version of Theorem 207. In fact,

$$4^{\log 7.5/\log 4} = \left(10^{\log 4}\right)^{\log 7.5/\log 4} = 10^{\log 7.5} = 7.5.$$

So, by Theorem 212, $\log_4 7.5 = \dfrac{\log 7.5}{\log 4}$.

A similar procedure proves:

Theorem 219.

$$\forall_{0 < a \neq 1} \forall_{0 < b \neq 1} \forall_{x > 0} \log_b x = \frac{\log_a x}{\log_a b}$$

which gives us the change-of-base formula.

Carry out the proof along the lines indicated above. [A slight variation on the proof goes this way:

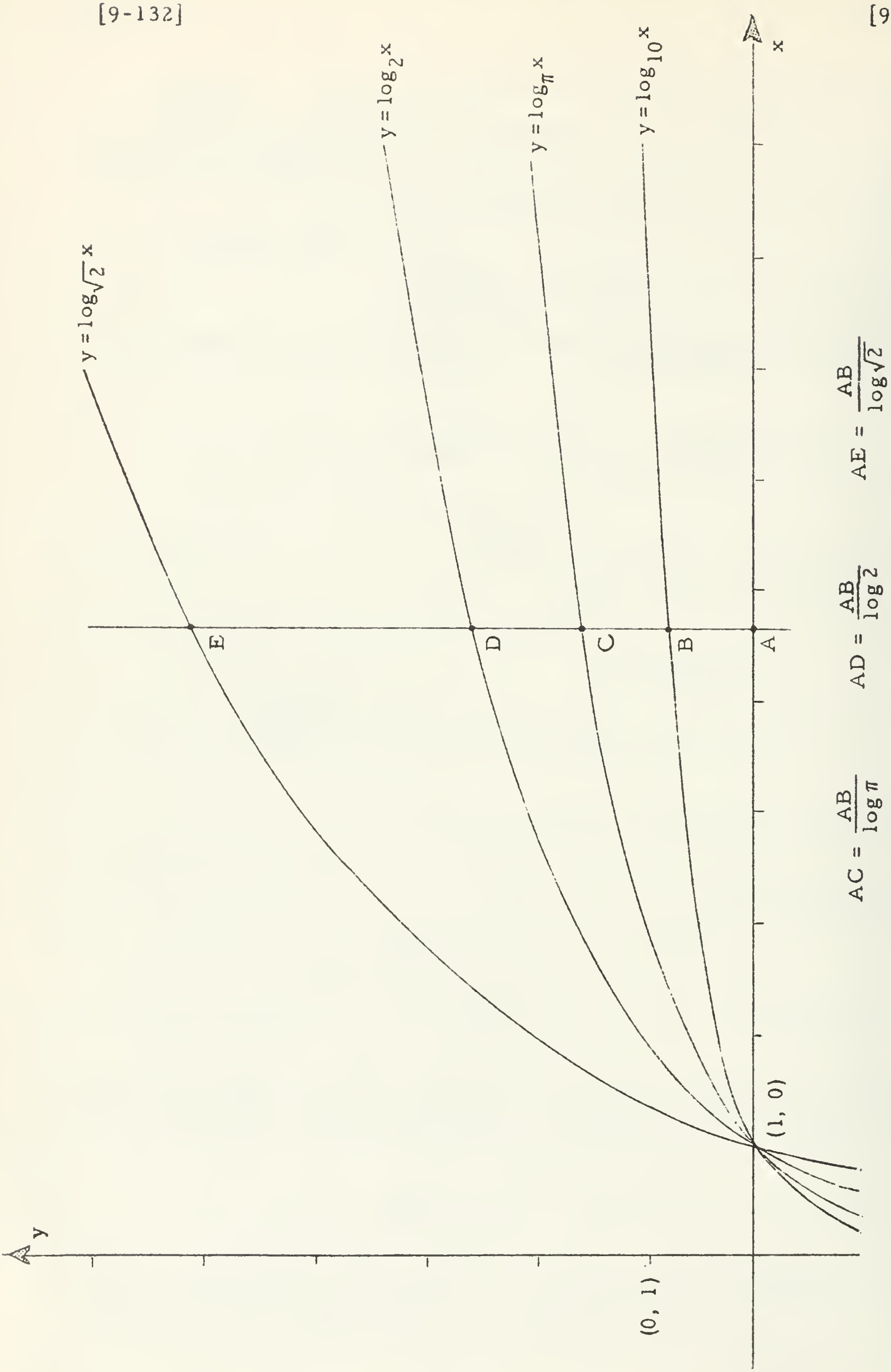
$$b^{\log_b c} = c; \text{ so, } \log_a\left(b^{\log_b c}\right) = \log_a c.$$

Hence [by Theorem 217], $\log_b c \cdot \log_a b = \log_a c$. Consequently, ...]

Using Theorem 218b [which is, in fact, an instance of Theorem 219], the formula of Theorem 219 can be rewritten:

$$\log_b x = \log_a x \cdot \log_b a$$

Since $a \neq 1$, $\log_b a \neq 0$. So, we see that each logarithm function can be obtained from any other by multiplying the latter by a nonzero constant. Conversely, each nonzero multiple of a logarithm function is, again, a logarithm function.


y = log√2 x
y = log2 x
y = logπ x
y = log10 x
x
y
E
D
C
B
A
(1, 0)
(0, 1)
AC = AB / log π
AD = AB / log 2
AE = AB / log √2

EXERCISES

A. 1. Prove the last statement made in the text. [Hint. Given $0 < a \neq 1$ and $c \neq 0$, for what number b, if any, is $c = \log_b a$? If you can find such a number b, is it positive and different from 1?]

2. Use a table of common logarithms to approximate each of the following:

(a) $\log_3 5$ (b) $\log_{13} 2$ (c) $\log_7 265$

(d) $\log_{98} 36$ (e) $\log_{0.5} 13$ (f) $\log_{1/3} 64$

3. Solve for 'x': $\log_b a \cdot \log_a c \cdot \log_c x = a$

B. Several logarithm curves are shown on page 9-132.

1. Why do all the curves pass through the point (1, 0)?

2. Fill in the blank so as to make the following sentence true.

For each positive abscissa, the ordinate to the $\log_{\sqrt{2}}$-curve is ______ the ordinate to the $\log_2$-curve.

3. Why is it the case, for each of the log curves drawn on page 9-132, that points on the curves which are to the right of (1, 0) are above the x-axis?

4. State a sentence like that in Exercise 2 but with '$\log_{0.5}$' in place of '$\log_{\sqrt{2}}$'.

5. Sketch, on page 9-132, the $\log_{0.5}$-curve.

C. The chart on page 9-132 illustrates the fact that all log curves have much the same shape. One can distinguish between two log curves by comparing how steep they are at (1, 0). Fill the blank in such a way as to make the following sentence true.

For each two numbers a and b, both greater than 1, the $\log_a$-curve is steeper at (1, 0) than the $\log_b$-curve if and only if ____________.

MISCELLANEOUS EXERCISES

1. Simplify.

(a) $\frac{a - 4}{a - 2} - \frac{a - 7}{a - 5}$

(b) $\frac{3t}{1 - t^2} - \frac{2}{t - 1} - \frac{2}{t + 1}$

2. Solve these systems.

(a) $\begin{cases} 4x + 7y = 29 \\ x + 3y = 11 \end{cases}$

(b) $\begin{cases} 9y + 10x - 290 = 0 \\ 12x - 11y - 130 = 0 \end{cases}$

3. Suppose that ship A is heading north at 12 knots and ship B is heading east at 9 knots. If A sights B 10 nautical miles directly north of A at 12 noon, at what time will they be the least distance apart?

4. Consider the semicircle $\overset{\frown}{AXB}$ with $\overleftrightarrow{AB}$ as diameter. Let N_1, N_2, N_3, ... be points of $\overline{AB}$ which divide it into congruent segments such that $AN_1 < AN_2 < AN_3 < \ldots$. Let P_1, P_2, P_3, ... be points of $\overset{\frown}{AXB}$ such that $\overleftrightarrow{P_1N_1}$, $\overleftrightarrow{P_2N_2}$, $\overleftrightarrow{P_3N_3}$, ... are perpendicular to $\overleftrightarrow{AB}$. Show that

$$\frac{AP_1^{\,2}}{1} = \frac{AP_2^{\,2}}{2} = \frac{AP_3^{\,2}}{3} = \ldots .$$

5. Factor.

(a) $28 - x - 2x^2$

(b) $6z^2 + 7z - 3$

(c) $m^2p^2 - 3mpq - 10q^2$

(d) $k^2j^2 - 16kj + 39$

(e) $4m^2 - 1$

(f) $1 - 49a^6$

(g) $y^4 - 9$

(h) $b^3 - by$

(i) $7r^2 + r$

(j) $s^6t^6 - 4$

6. Show that $\left(\frac{20}{21}\right)^{-100} > 100.$

7. Simplify.

(a) $\log \frac{75}{8} - 2 \log \frac{5}{9} + \log \frac{32}{243}$

(b) $\log \left[\left(\frac{7 \cdot 5^{-2}}{7^{-4} \cdot 5^3} \right)^{-3} \left(\frac{7^{-1} \cdot 5}{5^2 \cdot 7^{-3}} \right)^{-5} \right]$

8. Solve for 'x': $a^{x+1} \div b^{x-1} = c^{2x}$

9. Solve these systems.

(a) $\begin{cases} 9x^{-1} - 4y^{-1} = 1 \\ 18x^{-1} + 20y^{-1} = 16 \end{cases}$ (b) $\begin{cases} 5x^{-1} + y^{-1} = 79 \\ 16x^{-1} - y^{-1} = 44 \end{cases}$

10. Solve these systems for 'x' and 'y'.

(a) $\begin{cases} ax - by + cx = bc \\ x - a = b - y \end{cases}$ (b) $\begin{cases} xa^{-1} + yb^{-1} = c \\ xb^{-1} = ya^{-1} \end{cases}$

11. Solve the system: $\begin{cases} x^y = y^x \\ x^3 = y^2 \end{cases}$

12. Solve for 'x' and 'y': $\begin{cases} a^{3x} = m^{10}b^{2y} \\ a^{2x} = m^5b^{3y} \end{cases}$

13. Solve the system: $\begin{cases} 2^{a+b} = 6^b \\ 3^{a-1} = 2^{b+1} \end{cases}$

14. Simplify.

(a) $(3x^4 - 10x^3y + 22x^2y^2 - 22xy^3 + 15y^4) \div (x^2 - 2xy + 3y^2)$

(b) $(x^7 - 5x^5 + 7x^3 + 2x^2 - 6x - 2) \div (1 + 2x - 3x^2 + x^4)$

15. Solve these equations.

(a) $8(x - 1) - 17(3 - x) = 4(4x - 9) + 4$

(b) $8(x - 3) - 2(3 - x) = 2(2 + x) - 5(5 - x)$

(c) $5a - (3a - 7) - [4 - 2a - 3(2a - 1)] = 10$

16. Consider a convex quadrilateral one of whose diagonals bisects the region bounded by the quadrilateral. Prove that this diagonal bisects the other diagonal.

17. If, in $\triangle ABC$, $AC = 8$ and $\angle A$ is an angle of $30°$, find the measure of the altitude $\overleftrightarrow{CD}$.

18. Suppose that $A = \sqrt{\frac{3B}{D}}$. If A is multiplied by 9 and D is a constant, what change takes place in B?

19. What is the greatest common divisor of 2160, 1344, and 1440?

20. If a first number is five times a second then $\frac{1}{8}$ of the second is ______% of the first.

21. Suppose that the ratio of the length-measure to the width-measure of a rectangle is 4 to 1. If the rectangle has the same area-measure as that of a square with side-measure 8, what are the dimensions of the rectangle?

22. Expand.

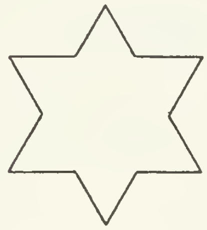
(a) $(3x^4 - 2x^3 + x - 7)(3x^2 - x + 1)$

(b) $(5a^3b^3 - 3a^2b^2 + 2ab - 1)(2a^2b^2 + ab - 5)$

23. If 360 is divided by a certain number, this divisor will be 1 more than the quotient and 1 more than the remainder. Find the number.

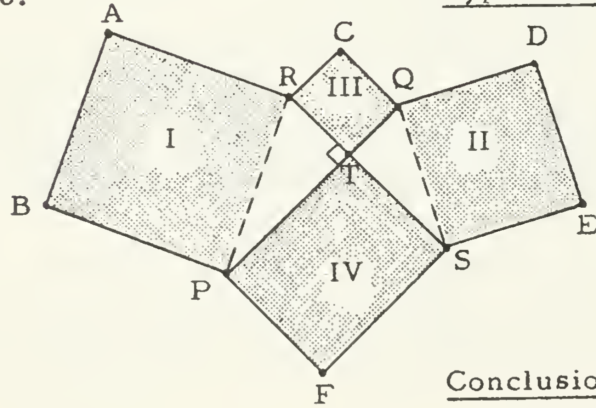
24. Solve the equation: $0.0000358 = 3.58 \times 10^x$

25.



The perimeter of this regular <u>hexagram</u> is 120. What is the area-measure of the region it bounds?

26.



<u>Hypothesis</u>: $\overleftrightarrow{QP} \perp \overleftrightarrow{RS}$ at T,

$T \in \overline{QP} \cap \overline{RS}$,

I and II are square regions,

III and IV are rectangular regions

<u>Conclusion</u>: $K_I \times K_{II} \geq (K_{III} + K_{IV})^2$

27. Suppose that O is the sequence of odd positive integers and E is the sequence of even positive integers. Prove:

(a) $\sum_{p=1}^{n} O_p + \sum_{p=1}^{n} E_p = \sum_{p=1}^{2n} p$ (b) $\sum_{p=m}^{n} O_p = n^2 - (m-1)^2$

(c) $\sum_{p=m}^{n} E_p = \sum_{p=1}^{n} E_p - \sum_{p=1}^{m-1} E_p$

28. If $a = -2$ and $b = 3$ then $a^{-2}b - ab^{-2} =$

29. If $a = 16$, $b = 8$, $c = 4$, and $d = 2$ then $(a - b)(cd)^{-1} =$

EXPLORATION EXERCISES

Consider the tripling function. Is the triple of the sum of two numbers the sum of their triples? Since the answer to this question is 'yes', the tripling function satisfies [that is, is a solution of] the open sentence:

(1) $$\forall_x \forall_y \; f(x + y) = f(x) + f(y)$$

In other words, if you replace 'f' in (1) by a name for the tripling function, the resulting statement is a theorem.

To prove this theorem, suppose that

$$g(x) = 3x, \text{ for all } x$$

--that is, suppose that g is the tripling function.
Then, by definition,

$$g(a + b) = 3(a + b),$$

and, since $g(a) = 3a$ and $g(b) = 3b$,

$$g(a) + g(b) = 3a + 3b.$$

Since, by the ℓdpma,

$$3(a + b) = 3a + 3b,$$

it follows that

$$g(a + b) = g(a) + g(b).$$

Consequently,

$$\forall_x \forall_y \; g(x + y) = g(x) + g(y).$$

So, g satisfies (1).

Here are some open sentences which are satisfied by various functions.

$$(1)\qquad \forall_x \forall_y\ f(x+y) = f(x) + f(y)$$

$$(2)\qquad \forall_{x>0} \forall_{y>0}\ f(xy) = f(x) + f(y)$$

$$(3)\qquad \forall_x \forall_y\ f(x+y) = f(x)f(y)$$

$$(4)\qquad \forall_x \forall_y\ f(xy) = f(x)f(y)$$

$$(5)\qquad \forall_{x>0} \forall_{y>0}\ f(xy) = f(x)f(y)$$

Does the tripling function satisfy (2)? Let's see. For $a > 0$ and $b > 0$, $g(ab) = 3ab$, $g(a) = 3a$, and $g(b) = 3b$. Is it the case that, for <u>each</u> $a > 0$ and <u>each</u> $b > 0$, $3ab = 3a + 3b$? No. So, the tripling function does not satisfy (2).

Does the tripling function satisfy (3)? Let's check. Is it the case that, for each a and each b, $3(a + b) = (3a)(3b)$? No. So, it doesn't satisfy (3). Does it satisfy (4)? [If it did satisfy (4), would it satisfy (5)?] Does it satisfy (5)?

<u>A</u>. You have seen that the tripling function satisfies (1). For each of (2), (3), (4), and (5), find a function which satisfies it.

<u>B</u>. In each of the following exercises you are given a function. For each function, tell which one(s) of the five sentences it satisfies.

1. the doubling function
2. $h(x) = x^2$
3. log
4. $\log_2$
5. the linear function defined by '$y = 7x + 3$'
6. $y = \pi x$
7. $\{(x, y)\colon y = 2\}$
8. the constant function 1
9. 0
10. the identity function [$y = x$]
11. $g(x) = \pi^x$
12. $h(x) = x^\pi$, for $x \geq 0$
13. the reciprocating function
14. the exponential function with base 3
15. $y = \sqrt{x}$
16. $y = (1/2)^x$
17. $\{(u, v)\colon v = 2^{-u}\}$
18. $y = 0^x$
19. $y = \log(3^x)$
20. $y = (\log 3)^x$
21. $y = 1^x$

22. a function f such that, for some $c \neq 0$, $f(x) = cx$

23. $\{(u, y): y = a^u\}$, where $0 < a \neq 1$

24. $\{(x, y), x > 0: y = x^c\}$, for some c different from 0 and 1

25. the function $g \circ \log$, where g is a function which satisfies (1).

C. Consider the function g which is defined for all real numbers as follows. For each x, if there are rational numbers r and s such that $x = r + s\sqrt{2}$ then $g(x) = r$; otherwise, $g(x) = 0$. So, for example,

$$\begin{aligned} g(5) &= g(5 + 0\sqrt{2}) = 5, \\ g(3 - \sqrt{2}) &= g(3 + {-1}\sqrt{2}) = 3, \\ g(0) &= g(0 + 0\sqrt{2}) = 0. \end{aligned}$$

1. Show that $g(\sqrt{3}) = 0$. [Hint. If there were rational numbers r and s such that $r + s\sqrt{2} = \sqrt{3}$ then $(r + s\sqrt{2})^2 = \ldots$.]

2. Compute $g(1 + \sqrt{3})$.

3. Does g satisfy (1)?

4. Does g have a subset--say, f--such that

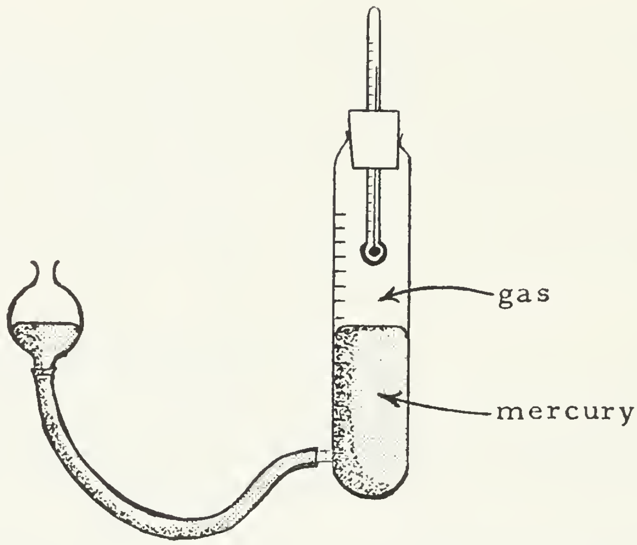
$$\forall_{x \in \mathfrak{D}_f} \forall_{y \in \mathfrak{D}_f}\ f(x + y) = f(x) + f(y)?$$

9.10 Some laws of nature.--In this section you will see one way in which mathematics helps in discovering physical laws. In the process you will learn more mathematics and become acquainted with a number which crops up in many parts of mathematics and in many of its applications. In fact, this number is so ubiquitous that, like π, it has been given a letter name, 'e'.

GAY-LUSSAC'S LAW FOR GASSES

In an earlier unit we talked about Boyle's Law for gasses which says that the volume V of a gas sample whose temperature is constant is inversely proportional to its pressure P. Gay-Lussac's Law describes the behavior of a gas sample whose pressure is constant. Let's see how such a law might be discovered.

A physicist wishes to determine whether the volume V of a gas sample whose pressure is kept fixed is a function of its temperature T and, if so, what function V is of T. To do this he might make use of an apparatus like that shown below. The pressure on the gas in the graduated tube can be kept fixed by raising or lowering the mercury reservoir



so that the difference in the levels of the two surfaces of the mercury remains the same. The volume of the sample is read from the graduations on the tube, and the thermometer gives the temperature of the gas in degrees centigrade. A series of experiments is then carried out. The physicist varies the temperature of the gas sample by applying a flame to the tube or packing the tube in some cooling agent. Each reading of temperature and volume gives a pair (t, v) of numbers.

Here is a table of some of the ordered pairs he obtains:

t	-4	20	16	76	44	48	28	16	60	12	52	84	28
v	49.3	53.7	52.9	63.9	58.1	58.8	55.1	52.9	61.0	52.2	59.5	65.4	55.1

The physicist finds that any two experiments which yield the same value for the temperature also yield the same value for the volume--more correctly, the differences are small enough to be laid to experimental errors [errors in adjusting the height of the mercury reservoir or in making the two readings]. So, he concludes that the volume is, indeed,

a function of the temperature--that is, that there is a function g such that

$$V = g \circ T.$$

In order to discover the function g, he again examines his data. What he finds is that, to within experimental errors, whenever the difference in the temperatures he measured in two experiments is the same as the difference in temperatures for another two experiments, it is also the case that the differences in volumes for the two pairs of experiments are the same--changing the temperature by a given number of degrees always results in the same change in volume, no matter what temperature he starts from [as long as the pressure is kept constant].

Now, what this means is that there is a function f such that, for any two temperatures t_1 and t_2 and corresponding volumes $g(t_1)$ and $g(t_2)$,

$$(1) \qquad g(t_2) - g(t_1) = f(t_2 - t_1).$$

In particular, taking t_1 to be 0 and replacing 't_2' by 'x',

$$(2) \qquad \forall_x\, f(x) = g(x) - g(0).$$

Consequently, if we can find what the function f is then, knowing the volume g(0) of the gas sample at 0°C, we can find the function g and, so, learn how the volume of the sample depends on its temperature.

To discover f we begin by noting that, by (2), for any t_1 and t_2,

$$\begin{aligned} f(t_2) - f(t_1) &= [g(t_2) - g(0)] - [g(t_1) - g(0)] \\ &\left.\begin{aligned} &= g(t_2) - g(t_1) \\ &= f(t_2 - t_1), \end{aligned}\right\} (1) \end{aligned}$$

that is, $$f(t_2) = f(t_2 - t_1) + f(t_1).$$

This result becomes more illuminating if we replace '$t_2 - t_1$' by 'x', 't_1' by 'y', and, of course, 't_2' by '$x + y$'. We then see that f must satisfy:

$$(3) \qquad \forall_x \forall_y\, f(x + y) = f(x) + f(y).$$

The generalization (3) may suggest to you the ℓdpma. In fact, if f is a function such that, for some c,

$$(4) \qquad f(x) = cx$$

then, by the ℓdpma, f satisfies (3). [Such a function f for which $c \neq 0$ is

called a homogeneous linear function.] Since the physicist's function f cannot well be the constant function whose value is 0, this suggests the idea that the physicist's function f may be a homogeneous linear function. We could be sure that this is the case if we knew that each function which satisfies (3) is a homogeneous linear function. [Notice that this is quite different from what we just discovered--that each homogeneous linear function does satisfy (3).] As a matter of fact, (3) has many solutions which are not homogeneous linear functions. However, as is shown in Appendix E [Theorem 221], except for the constant function whose value is 0, these extra solutions are very queer functions indeed. Each such function f has the surprising property that, given any ordered pair of real numbers, there is an ordered pair (x, f(x)) which belongs to the function and is as close to the given ordered pair as you wish. Now, such a function--one which contains ordered pairs arbitrarily close to each ordered pair in the whole number plane--certainly cannot be the physicist's function. [For one thing, as the physicist may observe, his function is an increasing one, and none of these queer solutions of (3) can be as simple as that.] So, since each solution of (3) is either a homogeneous linear function or a "physically impossible" function, the physicist's function f must be linear and homogeneous.

Returning now to the function g, we see from (2) and (4), that, for any temperature t,

$$\begin{aligned} g(t) &= g(0) + f(t) \\ &= g(0) + ct, \end{aligned}$$

for some number $c \neq 0$. The more usual form of writing this is:

$$g(t) = g(0)[1 + \alpha t]$$

in which, of course, $\alpha = c/g(0)$. Since g is the function such that

$$V = g \circ T,$$

we see that, if v_0 is the volume-measure of the gas at 0°C then

$$(5) \qquad V = v_0[1 + \alpha T]. \qquad \text{[Gay-Lussac's Law]}$$

[Surprisingly enough, it turns out that the constant α is nearly the same for all gas samples--no matter what size the sample is, no matter what kind of gas, and no matter at what value the pressure is fixed, the constant α is approximately 0.00367.]

EXERCISES

A. A certain gas sample has a volume of 120 cubic centimeters at a temperature of 0°C. If, without changing the pressure, the temperature is reduced to −273°C, what is the new volume of the sample? [Actually, any gas sample will liquefy at sufficiently low temperatures and, being no longer a gas sample, Gay-Lussac's Law will not apply. Since $1/\alpha \doteq 273$, any gas sample which remained a gas at about −273°C would have volume-measure 0! This, and other evidence, suggests that nothing can be colder than −273°C (approximately), and this temperature is called absolute zero. The absolute temperature of an object is its centigrade temperature plus 273.]

* * *

According to Gay-Lussac's Law, if the pressure of a gas sample is held constant and its volume at 0°C [at the given pressure] is v_0 then its volume v at t°C is given by the formula:

$$v = v_0[1 + \alpha t]$$
$$= v_0\alpha[t + \frac{1}{\alpha}]$$

By definition [see above], $t + \frac{1}{\alpha}$ is the absolute temperature. So, Gay-Lussac's Law says that, under isopiestic conditions [fixed pressure], the volume V of a gas sample is proportional to the absolute temperature T. Of course, the constant of proportionality depends on the given pressure. Hence, what Gay-Lussac's Law tells us is that, for each gas sample, there is a function g such that

$$(1) \qquad V = (g \circ P) \cdot T.$$

[For each pressure p_0, $g(p_0) = v_0\alpha$.]

Now, according to Boyle's Law, under isothermal conditions [fixed temperature], P is inversely proportional to V. In this case, the constant of proportionality depends on the fixed temperature. Hence, Boyle's Law tells us that, for each gas sample, there is a function b such that

$$(2) \qquad PV = b \circ T.$$

Let's see if we can find out what function b is.

Since, from (1),

$$(3)\qquad PV = P\cdot[(g\circ P)\cdot T] = [P\cdot(g\circ P)]\cdot T,$$

it follows from (2) and (3) that

$$b\circ T = [P\cdot(g\circ P)]\cdot T,$$

that is, that

$$(4)\qquad (b\circ T) \div T = P\cdot(g\circ P).$$

Now, (4) tells us a surprising thing. Choose some value of T. According to (2), we can get any value of P we like by varying V. By fixing a value of T, we have fixed the value of the function $(b\circ T) \div T$. So, the function $P\cdot(g\circ P)$ is some constant function, say, c. Hence, from (3),

$$(5)\qquad PV = cT. \qquad \text{[General Gas Law]}$$

[Of course, the function $(b\circ T) \div T$ is also a constant function. Why? So, what kind of a function is b?]

Since for different gas samples we have different functions g and b [for example, looking at (1) we can see that for a sample containing twice as much gas as another, the function g for the first sample is twice the function for the second], it is reasonable to expect that the function c will vary from one sample to another. This is the case, but interestingly enough, c depends only on the number of molecules in the sample and not on the kind of gas. In fact, c is proportional to the number N of molecules in the sample. If the values of P and V are in metric units [dynes per square centimeter and cubic centimeters] then the constant of proportionality is approximately 1.38×10^{-16}. So,

$$(6)\qquad PV \doteq 1.38 \times 10^{-16}\, NT.$$

It should be remarked that while Gay-Lussac's Law holds with considerable accuracy over a large range of volumes and temperatures, the same cannot be said for Boyle's Law. For most gasses, the latter holds only for moderate ranges of values of P, V, and T. So, the same applies to (6).

<u>Example.</u> The volume of a gas sample is 137.9 cubic inches at 294°C and a pressure of 28.6 pounds per square inch. What is its volume at 137°C and a pressure of 35.6 pounds per square inch?

Solution. In view of formula (5), we see that

$$\frac{p_2 v_2}{p_1 v_1} = \frac{t_2}{t_1}.$$

So,

$$\frac{35.6 \times v_2}{28.6 \times 137.9} = \frac{137 + 273}{294 + 273}.$$

Hence,

$$v_2 = \frac{28.6 \times 137.9 \times 410}{35.6 \times 567}.$$

By logarithms,

$$v_2 \doteq 80.12.$$

So, the volume is about 80.12 cubic inches.

* * *

B. Solve the following problems.

1. A gas sample under a pressure of 38.2 pounds per square inch has a volume of 183.5 cubic inches at a temperature of 86.3°C. What is its volume under a pressure of 46.3 pounds per square inch and at a temperature of 47.5°C?

2. The pressure on a gas sample is doubled and its absolute temperature is doubled. What change takes place in the volume?

3. If the pressure on a gas sample is increased by 25% and the temperature is kept fixed, what is the %-change in the volume?

4. If the pressure on a gas sample is increased by m% and the absolute temperature is decreased by n%, what is the %-change in the volume?

5. A gas sample under a pressure of 43.6 pounds per square inch has a volume of 216.3 cubic inches at a temperature of 74.3°C. What is its volume under a pressure of 53.6 pounds per square inch and at a temperature of 74.3°C?

6. Suppose that the temperature of a gas sample is increased from 50°C to 75°C without a change in pressure. What is the %-change in the volume?

7. A gas sample had, at one time, a pressure of 15 pounds per square inch and a volume of 1 cubic foot. At a later time its pressure was 20 pounds per square inch and its volume was 1.5 cubic feet.

   (a) At which time did the gas have the higher temperature?

   (b) What is the ratio of the absolute temperature-measures of the gas at the two times?

   (c) What is the ratio of the centigrade temperature-measures of the gas at the two times?

   (d) If the temperature of the gas at the earlier time was $-10°C$, what was its centigrade temperature at the later time?

C. Pretend that someone is trying to discover what function the perimeter P is of the length-measure L of a rectangle of constant width. He accumulates some data:

ℓ	15	13	18	30	25	23	13	18
p	50	46	56	80	70	66	46	56

He knows that there is a function g such that

$$P = g \circ L.$$

Examining the data further leads him to suggest that the difference in perimeters is a function, f, of the difference in corresponding length-measures. That is, for any $\ell_2 > 0$ and $\ell_1 > 0$,

$$g(\ell_2) - g(\ell_1) = f(\underline{\quad ? \quad}).$$

In particular, taking ℓ_1 to be, say, 30,

$$(*) \qquad \forall_{x>0}\ f(x - 30) = g(x) - \underline{\quad ? \quad}.$$

Now, to discover f, he notes that, for any $\ell_2 > 0$ and $\ell_1 > 0$,

$$\begin{aligned} f(\ell_2 - 30) - f(\ell_1 - 30) &= [g(\ell_2) - \underline{\ ? \ }] - [g(\ell_1) - \underline{\ ? \ }] \\ &= g(\ell_2) - \underline{\ ? \ } \\ &= f(\underline{\quad ? \quad}), \end{aligned}$$

that is, $f(\ell_2 - 30) = f(\underline{\quad ? \quad}) + f(\ell_1 - 30)$

Replacing '$\ell_2 - \ell_1$' by 'x' and '$\ell_1 - 30$' by 'y', and, of course, '$\ell_2 - 30$' by '$\underline{\quad ? \quad}$', we see that f must satisfy:

$$\forall_{x > -30} \forall_{y > -30}\ f(\underline{\quad ? \quad}) = f(\underline{\ ? \ }) + f(\underline{\ ? \ })$$

Aside from certain queer functions, the only functions which meet this condition are subsets of the constant function 0 and of linear homogeneous functions. From the nature of the problem [for example, f is an increasing function], we can rule out the queer functions and 0. Hence, f is a subset of a homogeneous linear function. So, there is a number $c \neq 0$ such that

$$\forall_{x > -30}\ f(x) = \underline{\quad ? \quad}.$$

Substituting in (*), we have:

$$\forall_{x > 0}\ c \cdot (\underline{\quad ? \quad}) = g(x) - \underline{\quad ? \quad}$$

In other words, for some number c, and for any length-measure $\ell > 0$,

$$(**) \qquad g(\ell) = \underline{\quad ? \quad} + c \cdot (\underline{\quad ? \quad}).$$

We can tell what number c is by using any ordered pair from the table. Take the pair (15, 50), and substitute in (**):

$$50 = \underline{\quad ? \quad} + c \cdot (15 - 30)$$

Hence, $c = \underline{\ ? \ }$, and g is the function such that, for each $\ell > 0$,

$$g(\ell) = 20 + \underline{\ ? \ }.$$

Since $P = g \circ L$, we see that

$$P = \underline{\qquad ? \qquad}.$$

DECAY OF A RADIOACTIVE SUBSTANCE

Now, let's see how a physicist might tackle the problem suggested in the Preface [page 9-iii]. There it was said that a sample which, at one time, contains A(0) grams of a certain isotope of strontium will, t years later, contain A(t) grams of the isotope, where

$$A(t) = A(0) \cdot 10^{-2t}.$$

[The reason for the loss of weight is that some of the atoms of the

strontium isotope have "decayed" into less heavy atoms of another element.] Our problem is to see how the above formula might be discovered.

To begin with, the physicist needs some experimental data. For a start he may collect several samples from the same source and make a chemical analysis of some of them. This analysis can tell him how much strontium isotope he has to start with in one of the remaining samples. Suppose that this is $A(0)$ grams at time 0. Then, at various times t, he can, from the weight of the sample, calculate the weight of the strontium isotope still left in the sample at these times t. In this way he builds a table of approximations to values of the function A for various arguments.

In order to discover what the function A is, the physicist examines the data he has accumulated. In doing so he discovers that, to within experimental error, whenever the time interval between two weighings is the same as that between two others, the ratio of the weights of strontium isotope for the first two times is the same as the ratio for the other two--during all intervals of the same duration, there is the same percentage weight loss in the strontium isotope.

Now, what this means is that there is a function f such that, for any times t_1 and t_2, with $t_2 \geq t_1 \geq 0$,

$$(1) \qquad \frac{A(t_2)}{A(t_1)} = f(t_2 - t_1).$$

In particular, taking t_1 to be 0 and replacing 't_2' by 'x',

$$(2) \qquad \forall_{x \geq 0}\ f(x) = A(x)/A(0).$$

Consequently, if we can find the function f then, knowing $A(0)$, we can find the function A.

To discover f we begin by noting that, by (2), for $t_2 \geq t_1 \geq 0$,

$$\frac{f(t_2)}{f(t_1)} = \frac{A(t_2)/A(0)}{A(t_1)/A(0)}$$

$$\left.\begin{aligned} &= \frac{A(t_2)}{A(t_1)} \\ &= f(t_2 - t_1), \end{aligned}\right\}(1)$$

that is, $$f(t_2) = f(t_2 - t_1)f(t_1).$$

It is convenient to replace, here, '$t_2 - t_1$' by 'u', 't_1' by 'v', and, of course, 't_2' by '$u + v$'. Doing so, we see that f is a function such that

$$(3) \qquad \forall_{u \geq 0} \forall_{v \geq 0}\ f(u + v) = f(u)f(v).$$

The generalization (3) may suggest to you Theorem 205:

$$\forall_{x > 0} \forall_u \forall_v\ x^{u+v} = x^u x^v$$

and, so, suggest that the function f is an exponential function restricted to nonnegative arguments. This is, in fact, the case. To prove it, we make use of the result on homogeneous linear functions described on page 9-142. We begin by noting that if f is any function which satisfies (3) then, for any $u \geq 0$,

$$f(u) = f(\tfrac{u}{2} + \tfrac{u}{2}) = [f(\tfrac{u}{2})]^2 \geq 0,$$

and since, in particular, $f(0) = [f(0)]^2$, either $f(0) = 0$ or $f(0) = 1$. Also, if $f(0) = 0$ then, for any $u \geq 0$,

$$f(u) = f(u + 0) = f(u)f(0) = f(u) \cdot 0 = 0$$

--that is, f is the constant function whose value is 0 and whose domain is $\{x: x \geq 0\}$. Using a slightly more complicated argument [for which, see Appendix E] it can be shown that if $f(0) = 1$ then either f is the exponential function with base 0 or, for each $u \geq 0$, $f(u) > 0$. To discover what f may be in this last case we consider the function F where $F = \log \circ f$. By (3) [since, for each $u \geq 0$, $f(u) > 0$], it follows that, for each $u \geq 0$ and $v \geq 0$,

$$\begin{aligned} \log (f(u + v)) &= \log (f(u)f(v)) \\ &= \log (f(u)) + \log (f(v)) \end{aligned}$$

--in other words,

$$(4) \qquad \forall_{u \geq 0} \forall_{v \geq 0}\ F(u + v) = F(u) + F(v).$$

Now, just as in the case of (3) on page 9-141, it can be proved that any function F whose domain is $\{u: u \geq 0\}$ and which satisfies (4) is either a homogeneous linear function, or the constant function 0, restricted to nonnegative arguments, or a very queer function--so queer that there are pairs $(u, F(u))$ belonging to the function which are arbitrarily close

to any given ordered pair whose first component is nonnegative. Since, whatever F is,

$$f(u) = 10^{F(u)} \qquad [\text{for } u \geq 0],$$

it follows that, in the case we are considering, either there is a number c such that,

$$(5) \qquad f(u) = 10^{cu} \qquad [\text{for } u \geq 0]$$

or there are pairs (u, f(u)) arbitrarily close to any point in the first quadrant.

Summarizing, the only functions which satisfy **(3)**, above, are these:

(i) the constant function whose value is 0 and whose domain is $\{x: x \geq 0\}$,

(ii) the exponential function with base 0,

(iii) the exponential functions with base $[10^c]$ greater than 0 restricted to $\{x: x \geq 0\}$,

and (iv) some very queer functions.

Since physical considerations show that the function f which we are seeking [see (3), above] cannot be one of those described under (i), (ii), or (iv), it must be an exponential function with positive base. So, by (3) and (5),

$$(6) \qquad \forall_{t \geq 0}\ A(t) = A(0) \cdot 10^{ct},$$

for some number c. The number c can be determined if we know the value of A for some argument in addition to 0. For example,

$$c = \log (A(1)/A(0)). \qquad [\text{Explain.}]$$

Measurements made on strontium isotope samples of the kind referred to in the Preface give $c \doteq -2$ [when, as we have been assuming, time is measured in years].

EXERCISES

A. The function A such that, for all $t \geq 0$,

$$(*) \qquad A(t) = A(0) \cdot 10^{-2t}$$

describes the decay of a certain strontium isotope. It tells us that if we start with a sample containing A(0) grams of this isotope,

there will be $A(0) \cdot 10^{-2t}$ grams left at the end of a t-year period.

1. The value -2 of 'c' was determined on the basis of using one year as the unit of measurement for time. What is the value of 'c' if the time-unit is one day? One century?

2. How many years will it take for a sample containing this strontium isotope to decay to a sample which contains just half as much of the isotope? [Hint. Suppose that x is the required number of years. Then, $A(x) = [2 \cdot A(x)] \cdot 10^{-2x}$. So, ...]

*

Since radioactive substances do decay in the manner described by (6), one can record the facts concerning the decay of such substances by merely listing, for each substance, the experimentally-determined value of 'c'. A more common practice is to give the time required for half the isotope to decay. This is called <u>the</u> <u>half-life</u> of the substance in question.

*

3. In terms of 'c', what is the number of years in the half-life of any radioactive substance?

4. What is the half-life in days of the strontium isotope referred to in (*)?

5. The strontium isotope in Exercise 4 is Sr^{89}. Unfortunately, the strontium isotope which people worry about is Sr^{90} and <u>its</u> half-life is 25 years. What is its law of decay?

6. The half-life of uranium-232 is 70 years. How long will it take 3/4 of the U^{232} in a sample to decay?

7. The half-life of uranium-234 is 2.48×10^5 years. How long will it take 1/4 of the U^{234} in a sample to decay?

8. (a) Draw a graph of '$y = 10^{-2t}$', plotting points for t = 0, 0.1, 0.2, . . . , 1.

   (b) Relabel the vertical scale so that what you have is a graph of the decay law of Sr^{89}.

B. Solve these problems.

1. The half-life of radium-226 is 1620 years. What fraction of the Ra^{226} in a sample remains after 810 years?

2. Due to radioactive decay, the neptunium-237 content of a sample decreases by 75% in 4.4 million years. What is the half-life of Np^{237}?

3. One of the isotopes of silver decays to 75% of its initial amount in about a half-hour. What is its half-life?

ANOTHER APPROACH TO RADIOACTIVE DECAY

The appearance of the '10' in generalization (6) on page 9-150 [and in the earlier formula (5)] was purely fortuitous--had we, for example, defined the function F to be $\log_2 \circ f$ rather than $\log \circ f$, we would have found that, for $t \geq 0$, $A(t) = A(0) \cdot 2^{at}$, for some number a [$a = c/\log 2$. Why?]. The function A can, evidently, be thought of as the product of the constant A(0) and a function determined by composing the exponential function with any positive base different from 1 with a properly chosen homogeneous linear function restricted to nonnegative arguments. There seems no very compelling reason to prefer one base to another. [The use of 10 as a base might simplify computations and, for base 2, the number a is related in a particularly simple way to the half-life [How?]. But the fact that we use base-10 numerals and that we choose to speak of half-lives [rather than, say, third-lives] doesn't have much to do with the nature of radioactive decay.] However, approaching the problem of decay in a different way will lead us to discover a number which is, in a real sense, the most "natural" base to use. This number occurs in many parts of mathematics and its applications and is denoted by the letter 'e'. The function $\log_e$ is called <u>the natural logarithm function</u> [and '$\log_e$' is abbreviated 'ℓn'], and the exponential function with base e is called <u>the</u> exponential function. To discover what e is, let's consider what a second physicist might do when confronted by the first one's table of values of the function A.

To this second physicist it seems likely that, for time intervals of the same duration, say h, the number of atoms which decay should be

proportional to the number of undecayed atoms present at the beginning of the interval. So, if $N(u)$ is the number of undecayed atoms present at time u then the ratio

$$\frac{N(u) - N(u + h)}{N(u)}$$

should be the same for all $u \geq 0$. Since, also, A is proportional to N, it should be the case that the ratio

$$\frac{A(u) - A(u + h)}{A(u)}$$

is the same for all $u \geq 0$ [Explain.].

The physicist checks this conjecture, using the values of A listed in his table, and finds that, to within experimental errors, it is confirmed--so, there is a function g such that, for $h > 0$ and $u \geq 0$,

$$\frac{A(u) - A(u + h)}{A(u)} = g(h).$$

In checking his conjecture, the physicist has accumulated a table of values of g and finds, naturally enough, that g is an increasing function --the longer the sample sits around, the more of its atoms will have decayed. The physicist now makes another conjecture--that $g(h)$ is proportional to h. If this is correct, there should be a positive number λ ["lambda"] such that, for $h > 0$,

$$g(h) = \lambda h.$$

To check this, he adds a new line to his table in which he lists values of '$g(h)/h$'. This time, he is not so lucky. The entries are not the same. However, he does notice that, for small values of 'h', the values of '$g(h)/h$' are all fairly close to 4.6, and are closer the smaller the value of 'h'. So, he modifies his conjecture and guesses that, for small positive values of 'h',

$$g(h) \doteq \lambda h$$

where $\lambda \doteq 4.6$. Summarizing, his data suggest that, for small $h > 0$, and all $u \geq 0$,

$$(1) \qquad \frac{A(u) - A(u + h)}{hA(u)} \doteq \lambda \qquad [\doteq 4.6],$$

and that the error can be made as small as he wishes by taking h sufficiently small. [λ is called <u>the decay constant</u>.]

This suggests a way of computing approximations to the values $A(t)$

of A. For any $t > 0$, if n is a large positive integer then t/n is a small positive number and, choosing this for h, one can, knowing $A(0)$, use (1) with $u = 0$ to compute an approximation to $A(t/n)$. Then, using this approximation, one can take $u = t/n$ in (1) and again using t/n for h, compute an approximation to $A(2t/n)$. After n such steps, one would obtain an approximation to $A(nt/n)$, that is, to $A(t)$.

In order to do this, let's rewrite (1) as:

$$A(u + h) \doteq (1 - \lambda h)A(u)$$

and replace 'h' by 't/n' to get:

$$(2) \qquad A(u + \frac{t}{n}) \doteq (1 - \frac{\lambda t}{n})A(u)$$

For $u = 0$, this tells us that

$$A(\frac{t}{n}) \doteq (1 - \frac{\lambda t}{n})A(0).$$

Using (2) again, this time for $u = \frac{t}{n}$, we find that

$$A(\frac{2t}{n}) \doteq (1 - \frac{\lambda t}{n})A(\frac{t}{n})$$

$$\doteq (1 - \frac{\lambda t}{n})^2 A(0).$$

Since $t = nt/n$, repetition leads eventually to the formula:

$$(3) \qquad A(t) \doteq (1 - \frac{\lambda t}{n})^n A(0)$$

Of course, since (2) yields only approximations, its repeated use may result in rather large errors. However, the basic error can be reduced as much as one wishes by choosing a large enough value for 'n'. Although doing so will increase the number of steps and, so, might leave us as badly off as before, it's worth trying. Let's see what happens to the values of '$(1 - \frac{\lambda t}{n})^n$' as n gets larger [for given values of 'λ' and 't'].

To do this, it is convenient to transform the expression '$(1 - \frac{\lambda t}{n})^n$'. By the principle for subtraction and Theorem 207,

$$(1 - \frac{\lambda t}{n})^n = \left[\left(1 + -\frac{\lambda t}{n}\right)^{-\frac{n}{\lambda t}}\right]^{-\lambda t}.$$

So, since $-\frac{n}{\lambda t}$ is the reciprocal of $-\frac{\lambda t}{n}$,

$$(4) \qquad (1 - \frac{\lambda t}{n})^n = \left[(1 + x)^{1/x}\right]^{-\lambda t}$$

where $x = -\lambda t/n$. Since, for large n, $\lambda t/n$ is a small positive number,

our problem is to investigate the values of '$(1 + x)^{1/x}$' for negative values of 'x' close to 0. Here is a table of approximations to some values of the function f for which, for $x \neq 0$, $f(x) = (1 + x)^{1/x}$.

x	$-1/2$	$-1/3$	$-1/5$	-10^{-1}	-10^{-2}	-10^{-3}	-10^{-4}	-10^{-5}
f(x)	4	3.38	3.05	2.87	2.73	2.7196	2.7184	2.71829

The evidence is that the function f is a decreasing function and that the values of f for negative arguments have a greatest lower bound which is not much different from 2.718. More evidence for this is furnished by considering values of f for some positive arguments.

x	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
f(x)	2.71827	2.71815	2.717	2.70	2.59

Comparison of the values for the arguments -10^{-5} and 10^{-5} suggests that the greatest lower bound of the values for negative arguments is between 2.71827 and 2.71829. As a matter of fact, this is the case. The greatest lower bound of these values [which is, also, the least upper bound of the values of f for positive arguments] is approximately 2.718281828459. This is the number e which was mentioned earlier.

So, we have seen that, whatever λ and t may be, $(1 - \frac{\lambda t}{n})^n$ is the $(-\lambda t)$th power of a number which, for n sufficiently large, is as close to e as we wish. Since the power function with exponent $-\lambda t$ is continuous at e [see Appendix C, Theorem 220], we conclude that, for n sufficiently large, $(1 - \frac{\lambda t}{n})^n$ is as close to $e^{-\lambda t}$ as we wish. Looking back at (3), this suggests that

$$\forall_{t \geq 0}\ A(t) = A(0) \cdot e^{-\lambda t},$$

where, as we saw earlier, $\lambda \doteq 4.6$.

To check this with the original formula:

$$\forall_{t \geq 0}\ A(t) = A(0) \cdot 10^{ct} \qquad [c \doteq -2]$$

we need only check whether $e^{-4.6} \doteq 10^{-2}$ or, equivalently, whether $e^{4.6} \doteq 10^2$. Since $e \doteq 2.718$, $\log e \doteq 0.4343$. Since $4.6 \times 0.4343 = 1.99778 \doteq 2$, the formulas are in good agreement.

Example. Find the half-life [in years] of Sr^{89}.

Solution. The law of decay of Sr^{89} is given by the formula:

$$A(t) = A(0) \cdot e^{-4.6t}$$

We are looking for the t such that A(t) is half of A(0)--that is, for the t such that

$$(*) \qquad e^{-4.6t} = 0.5.$$

Using natural logarithms we see that

$$-4.6t = \ln 0.5 = -\ln 2.$$

[Recall that '$\ln$' is an abbreviation for '$\log_e$'.] From a table of natural logarithms we find that $\ln 2 \doteq 0.69315$. So,

$$t \doteq \frac{0.69315}{4.6} \doteq 0.15.$$

Hence, the half-life of Sr^{89} is about 0.15 years.

Of course, if you don't have a table of natural logarithms available, you can still solve the problem using common logarithms. From (*), it follows that $-4.6t \log e = \log 0.5 = -\log 2$. Solving this for 't' amounts to using common logarithms to compute $\ln 2$ and then dividing by 4.6. Explain.

EXERCISES

A. 1. Plot a graph of '$y = e^x$' from $x = -2$ to $x = 2$. [Hint. Compute approximations to e^{-2}, $e^{-1.5}$, e^{-1}, ..., $e^{1.5}$, e^2. For example, since $\log e^{-1.5} = -1.5 \log e \doteq -1.5 \log 2.718 \doteq -1.5 \times 0.4343 \doteq -0.6515 = 9.3485 - 10$, $e^{-1.5} \doteq 0.223$. Of course, a table of natural logarithms makes the job easier.]

2. Sketch a graph of $\ln$.

B. Use Theorem 219 and common logarithms to find approximations to values of $\ln$.

1. $\ln 2$	2. $\ln 10$	3. $\ln 20$	4. $\ln 1.35$
5. $\ln(e^2)$	6. $\ln 0.95$	7. $\ln 1$	8. $\ln(-2.4)$

C. Find approximate solutions.

1. $\ell n\ x = 0.7894$ 2. $\ell n\ y = 2.7643$ 3. $\ell n\ z = -1.4786$

D. 1. Two functions which occur frequently in applications of mathematics are the functions c and s for which

$$c(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad s(x) = \frac{e^x - e^{-x}}{2}.$$

Sketch graphs of c and s. [Hint. First sketch graphs of '$y = e^x$' and '$y = e^{-x}$' on the same chart.]

*

The graph you drew of the function c suggests, by its shape, a rope or chain whose ends are fastened to two posts. As a matter of fact, it can be proved that a rope or chain of uniform weight per unit length does take this shape when it hangs freely between two points. More precisely, with respect to properly chosen coordinate axes, "the equation of such a chain" is '$y = a \cdot c(x/a)$', for some $a > 0$. For this reason [see the Latin for 'chain'], the curve you drew is called a catenary. For numerically small arguments,

$$c(x) \doteq 1 + \frac{x^2}{2}.$$

Consequently, near its lowest point a catenary is very nearly a parabola. As a matter of fact, a chain which is loaded in such a way that the total load per unit horizontal distance is constant has the shape of a parabola.

*

2. Use elementary algebra to show that, for each x,

(a) $[c(x)]^2 - [s(x)]^2 = 1$, and

(b) $2 \cdot c(x)s(x) = s(2x)$.

☆3. Which of c and s has an inverse? For the one which does, show that its inverse is the function f such that

$$f(x) = \ell n\left|x + \sqrt{x^2 + 1}\right|.$$

[Hint. See Exercise 10 of Part H on page 9-130.]

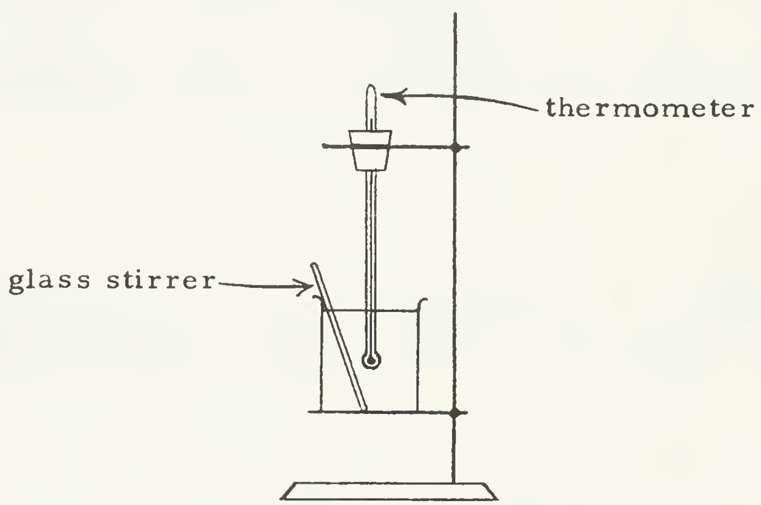
NEWTON'S LAW OF COOLING

As another example of exponential decay, consider the cooling of a warm body to room-temperature. If, at various times t, one measures the difference, T(t), between the temperature of the body and the [constant] temperature of the room, he finds that, to within experimental error, the decrease in T during an interval of short duration h is nearly proportional to the product of h by the value of T at the beginning of the interval. [Compare with (1) on page 9-153.] So, as in the case of radioactive decay, the law of cooling is such that, for some $k > 0$,

$$\forall_{t \geq 0}\ T(t) = T(0) \cdot e^{-kt}. \qquad \text{[Newton's Law]}$$

[See the derivation of (5) on page 9-155 from (1) on page 9-153.]

You can perform an interesting experiment in class or in the chemistry laboratory to check Newton's Law of Cooling. Heat some water to about 70°C and pour it into a beaker [about 250cc.] which is set on a ring-



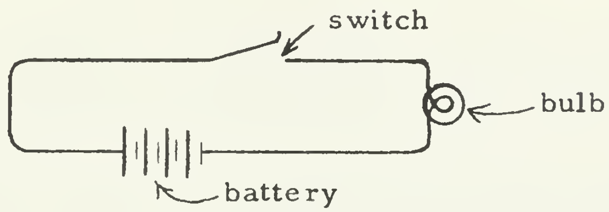
stand. Record the room-temperature. Then, make a temperature-time record in which you list water-temperatures at the ends of 1-minute intervals. Keep stirring the water with a glass rod while the water is cooling. Plot the difference between water-temperature and room-temperature against time. [T(0) is the temperature difference at the start, T(1) is the temperature difference at the end of the first minute of cooling, etc.] Do you get what looks like an exponential curve?

TRANSIENT CURRENTS IN SIMPLE CIRCUITS

The growth and decay of an electric current in certain simple circuits provide further examples of the occurrence of 'e' in statements of natural laws.

As a basis for understanding such laws, let's consider a situation in which the current is steady.

Consider a flashlight. Schematically, it looks like this:



The battery can exert a certain electromotive force in pushing electricity around. The measure of such a force is usually given in volts [after Volta, one of the earlier experimenters with electricity]--the cells in a 3-cell flashlight make up a battery of about 4 volts. When the switch is closed, the battery pushes electricity around the circuit--through the closed switch, through the lamp filament, back through the battery, and around again. [It may be helpful to think of the battery as a water-pump pushing water around a closed circuit of pipe. The switch is a valve, and the lamp filament is a rough section of pipe.] In pushing the electricity around, the battery has to overcome resistance [just as the water-pump has to work to overcome frictional resistance between the water and the wall of the pipe and the parts of the pump]. This resistance is usually measured in ohms, a unit named after the discoverer of Ohm's Law, who found that the rate at which electricity flows through the circuit is directly proportional to the electromotive force of the battery and inversely proportional to the resistance of the circuit. If E is the volt-measure of the electromotive force, and R is the ohm-measure of the resistance, then electricity flows at the rate of i amperes, where

$$i = \frac{E}{R}.$$

[Ampere, like Ohm and Volta, was a physicist who discovered many of the fundamental facts about electricity.] For example, a 6-volt battery can push electricity through a resistance of 3 ohms at a rate of 2 amperes.

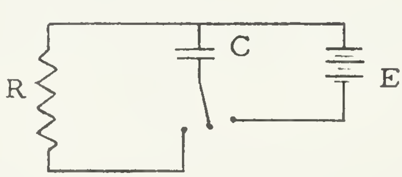
The unit used in measuring quantities of electricity is the coulomb--a rate of 1 ampere is 1 coulomb per second.

Now, let's consider another set-up. A condenser is one kind of device for storing electricity. In a simple case, a condenser may be a pair of metal plates which are fastened close together but not touching one another. Suppose that a battery is connected to the plates of a condenser, as indicated in the figure, by closing a switch. The battery

will pull electricity from one plate and push it onto the other. As it does this, its job gets harder. [Think of a water-pump lifting water from one reservoir and pushing it through a hole in the bottom of another. The higher the level of the water in the second reservoir gets--and the lower the level in the first reservoir--the harder it is for the pump to do its job. When the difference in levels becomes too great, the pump is no longer strong enough to lift any more water.] Finally, the battery will have moved as much electricity as it can, and nothing more happens. Coulomb discovered that the amount the battery can move is proportional to its electromotive force. The constant of proportionality, C, depends on the design of the condenser and is measured in farads [after Faraday] when the electromotive force E is measured in volts and the amount Q of electricity is measured in coulombs. With these units, Coulomb's Law is:

$$Q = CE$$

Suppose that we now disconnect the battery by opening the switch, and then switch the condenser into another circuit containing a resist-



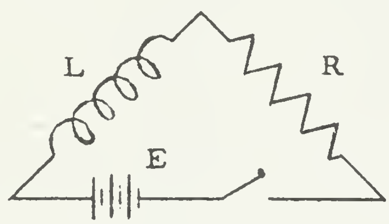
ance of R ohms. The electricity which has been stored on one plate of

the condenser will flow back through the resistance to the other plate. The rate at which current flows in the circuit will vary--"the higher the electricity is piled up, the faster it will flow away"--the less of the electricity which remains on the charged plate, the slower it will flow away. It can be shown that, t seconds after the switch is closed, the ampere-measured i_t of the current in the circuit is given by:

$$i_t = \frac{E}{R}e^{-\frac{t}{RC}}$$

Comparing this with Ohm's Law, we see that the discharge of the condenser creates a current like that due to a battery whose electromotive force t seconds after the switch is closed is $E\,e^{-t/(RC)}$.

Let's consider a third set-up. If current flows at an increasing rate through a coil of wire, there is generated in the coil a counter-electromotive force which tends to slow up the rate of flow of electricity. The amount of this counter-electromotive force is proportional to the rate at which the flow of current is changing. The constant of proportionality, L, measured in henries is called the self-inductance of the coil. Suppose a battery of E volts is connected to a resistance of R ohms and an inductance of L henries. At the instant the switch is closed, the cur-



rent in the circuit is 0 amperes. As this increases, a counter-electromotive force is set up in the coil which, effectively, reduces the electromotive force of the battery. So, the current cannot immediately build up, as it otherwise would, to E/R amperes. What happens is told by the formula:

$$i_t = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

From this we see that the current gradually builds up to the limiting value E/R which Ohm's Law leads us to expect.

COMPOUND INTEREST

Although this is not an example of a natural phenomenon, the growth of a "sample" of money on which interest is compounded bears some resemblance to exponential growth. Let's investigate this.

In this case, we don't have to do any experimental work to find the law of growth. By definition, a principal P invested for a given interest period at a simple interest rate k [that is, 100k% for the period] grows to an amount A such that

$$A = P + Pk = P(1 + k).$$

If you leave this amount A invested for a second period at the same rate k then the amount B due you at the end of the second period is

$$B = A(1 + k) = P(1 + k)^2.$$

In general, if you invest a principal A(0) for the duration t of m periods, the amount A(t) due you is

$$A(t) = A(0)(1 + k)^m.$$

Suppose that h is the length of one period. Then,

$$A(t + h) = A(0)(1 + k)^{m+1}.$$

So, the growth during this (m + 1)th period is

$$\begin{aligned} A(t + h) - A(t) &= A(0)[(1 + k)^{m+1} - (1 + k)^m] \\ &= A(0)(1 + k)^m[(1 + k) - 1] \\ &= A(t) \cdot k. \end{aligned}$$

Hence,

$$\frac{A(t + h) - A(t)}{A(t)} = k.$$

That is, the growth during an interest period is proportional to the amount at the beginning of the period. [Compare with (1) on page 9-153.]

Now, as it actually occurs, interest rates are given on an annual basis and the interest is compounded one or more times per year. If the annual interest rate is r and interest is compounded n times per year, then the interest period is 1/n years, the rate for that period is r/n, and the number of periods is nt. So, if $A_n(t)$ is the amount due at

the end of t years,

$$(1) \qquad A_n(t) = A_n(0)(1 + \frac{r}{n})^{nt}.$$

[Compare this with (3) on page 9-154.] As in the case of radioactive decay, for sufficiently small values of 'r/n',

$$(1 + \frac{r}{n})^{nt} \doteq e^{rt} \qquad \text{[Explain.]}.$$

So, for a given annual interest rate r,

$$(2) \qquad A_n(t) \doteq A_n(0) \cdot e^{rt}$$

for large values of 'n'.

The formula (2) is sometimes useful in getting an estimate of $A_n(t)$. The larger n is, the better is the estimate. To see how good an estimate this is, let's consider money invested at 6% for 5 years, compounded for various periods, and compare formulas (1) and (2).

Compounded	n	$(1 + \frac{0.06}{n})^{n5}$	$e^{0.06 \cdot 5}$	difference
annually	1	1.3382	1.3499	0.0117
semi-annually	2	1.3439	1.3499	0.0060
quarterly	4	1.3469	1.3499	0.0030
bimonthly	6	1.3478	1.3499	0.0021
monthly	12	1.3489	1.3499	0.0010
semi-monthly	24	1.3494	1.3499	0.0005

So, for example, the interest on \$100 at 6% compounded annually for 5 years is \$33.82. The approximation formula (2) gives you an estimate which is too large by \$1.17. But, if the interest were compounded semi-monthly, the estimate would be too large by only 5¢. If the interest were compounded weekly, by how much do you think the approximation formula would be in error?

The amount A(t) computed by the formula:

$$A(t) = A(0) \cdot e^{rt}$$

is said to be the amount A(0) grows to in t years at the annual rate r compounded <u>continuously</u>. If interest on an investment were compounded continuously at the annual rate of 6%, how many years would it take for the investment to double?

ADIABATIC COMPRESSION OF GASSES

As you may know from using a bicycle pump, compressing a gas heats it. Suppose that the outlet of the pump is closed so that, in effect, you have a closed cylinder with a piston. At the beginning of a stroke [see Figure 1], the pressure, volume, and absolute temperature have

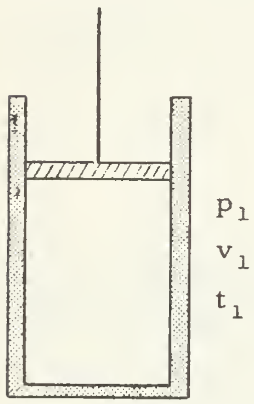


Fig. 1

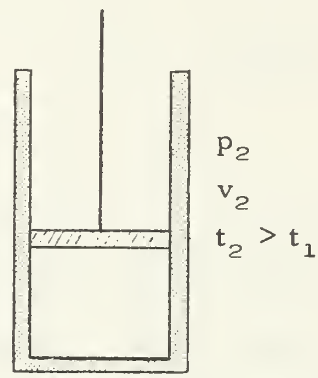


Fig. 2

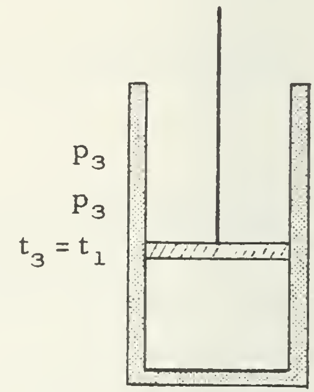


Fig. 3

given values p_1, v_1, and t_1. At the end of the stroke [see Figure 2], the values are p_2, v_2, and t_2. By the General Gas Law [see page 9-144],

$$\frac{p_2 v_2}{t_2} = \frac{p_1 v_1}{t_1} \qquad \text{[Explain.].}$$

Since compressing the gas raises its temperature, $t_2 > t_1$. So,

$$p_2 v_2 > p_1 v_1.$$

Now, suppose [see Figure 3] that you wait for the gas to cool down to the initial temperature t_1, holding the piston in the same position as you wait. How will the pressure p_3 compare with p_2? How will $p_3 v_3$ compare with $p_1 v_1$?

The answer to this last question is predicted by Boyle's Law which says that PV is a constant when T is a constant. That is, under isothermal conditions, PV is a constant. But consider the situation illustrated by Figure 2. Since $t_2 \neq t_1$, Boyle's Law doesn't apply. However, if the stroke is very fast so that there is little loss of heat from the gas during the stroke [or, better, if the cylinder is insulated against heat loss], there is another law which relates the initial and final pressures and volumes. Processes [compressions or expansions] during which the <u>heat content</u> [rather than the temperature] is constant are called <u>adiabatic</u>.

processes. Let's see how one might discover the adiabatic gas law.

When measurements are made of a gas sample under adiabatic conditions, one obtains pairs (p, v) of measures of pressure and volume. It turns out that if (p_1, v_1) and (p_2, v_2) are pairs of measures for which $v_1 = v_2$ then $p_1 = p_2$--that is, P is some function g of V. It also turns out that if (p_3, v_3) and (p_4, v_4) are other pairs of measures such that $v_4/v_3 = v_2/v_1$ then $p_4/p_3 = p_2/p_1$--that is, there is some function f whose value for the ratio of any two volume-measures is the ratio of the corresponding pressure-measures. [If Boyle's Law held, this f would be the reciprocating function.] Our problem is to find f and g.

Now, by the descriptions given above for f and g, it follows that, for any positive numbers [volume-measures] v_1 and v_2,

(1) $$f\left(\frac{v_2}{v_1}\right) = \frac{g(v_2)}{g(v_1)}.$$ [Compare with (1) on page 9-141.]

In particular, taking v_1 to be 1 and replacing 'v_2' by 'x',

(2) $$\forall_x\, f(x) = g(x)/g(1).$$

Consequently, if we can find what the function f is then, knowing the pressure g(1) when the volume of the gas sample is 1 unit, we can find g and, so, learn how the pressure of the sample depends on its volume.

To discover f we begin by noting that, by (2) and (1), for any positive v_1 and v_2,

$$\frac{f(v_2)}{f(v_1)} = \frac{g(v_2)/g(1)}{g(v_1)/g(1)} = \frac{g(v_2)}{g(v_1)} = f\left(\frac{v_2}{v_1}\right),$$

that is, $$f(v_2) = f\left(\frac{v_2}{v_1}\right) f(v_1).$$

Replacing '$\frac{v_2}{v_1}$' by 'x', 'v_1' by 'y', and 'v_2' by 'xy', we see that f must satisfy:

(3) $$\forall_{x>0}\,\forall_{y>0}\, f(xy) = f(x)f(y)$$

The generalization (3) may suggest to you Theorem 208:

$$\forall_{x>0}\,\forall_{y>0}\,\forall_u\, (xy)^u = x^u y^u$$

and, so, suggest that the function f is a power function restricted to positive arguments. This is, in fact, the case. Let's prove this.

Although there are many solutions of:

(3) $$\forall_{x>0}\forall_{y>0}\ f(xy) = f(x)f(y)$$

which are not power functions, none of these can fit the physical situation. To begin with, by inspection, the constant function 0 [defined on the positive numbers] is a solution of (3) and is not a power function. But, obviously, this constant function does not fit the physical conditions. Now, it is easy to see that each other solution f of (3) is such that, for each $x > 0$, $f(x) > 0$. For, to begin with, for $x > 0$,

$$f(x) = f(\sqrt{x}\cdot\sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) \geq 0.$$

Suppose now that, for some x_o, $f(x_o) = 0$. Then, for each $x > 0$,

$$f(x) = f\left(\frac{x}{x_o}\cdot x_o\right) = f\left(\frac{x}{x_o}\right)f(x_o) = f\left(\frac{x}{x_o}\right)0 = 0.$$

So, if f satisfies (3), either the values of f are all 0 or all positive.

In the latter case, (3) tells us that

(3′) $$\forall_{x>0}\forall_{y>0}\ \ln(f(xy)) = \ln(f(x)) + \ln(f(y)).$$

If in (3′) we had '$x + y$' instead of 'xy', (3′) would tell us [see Theorem 221] that $\ln\circ f$ is a subset of a homogeneous linear function [or a function too queer to fit the physical siutation]. Knowing what function $\ln\circ f$ is could enable us to find f. This suggests that we try to find another function F which is simply related to $\ln\circ f$ and such that

(*) $$\forall_x\forall_y\ F(x + y) = F(x) + F(y).$$

If, for $x > 0$ and $y > 0$, we let $a = \ln x$ and $b = \ln y$ then $x = e^a$ and $y = e^b$. It follows from (3′) that, for all a and b,

$$\ln(f(e^a e^b)) = \ln(f(e^a)) + \ln(f(e^b)).$$

But, $e^a\cdot e^b = e^{a+b}$. So,

$$\forall_a\forall_b\ \ln(f(e^{a+b})) = \ln(f(e^a)) + \ln(f(e^b)).$$

Comparing this result with (*), we see that if F is the function such that, for each a,

$$F(a) = \ln(f(e^a))$$

then F satisfies (*).

Since we know that f cannot be a queer function, neither can F. So, in view of (*), F is a homogeneous linear function or the constant function whose value is 0. Hence, for some c,

$$\ln(f(e^a)) = ca, \qquad \text{for all } a.$$

Since $x = e^a$ and $a = \ln x$, we have that, for some c,

$$\ln(f(x)) = c \ln x = \ln(x^c), \quad \text{for all } x > 0.$$

So, for some c,

$$f(x) = x^c, \qquad \text{for all } x > 0.$$

In other words, f is a power function restricted to positive arguments.

In the case of adiabatic processes, the exponent of the power function f turns out to be a negative number, $-\gamma$. [For air, $\gamma \doteq 1.4$.] So, returning to (1), we see that if (p_1, v_1) and (p_2, v_2) are pressure-volume measurements made on the sample under adiabatic conditions then

$$\frac{p_2}{p_1} = \frac{g(v_2)}{g(v_1)} = f\left(\frac{v_2}{v_1}\right) = \left(\frac{v_2}{v_1}\right)^{-\gamma} = \frac{{v_1}^{\gamma}}{{v_2}^{\gamma}},$$

that is,

$$p_2 {v_2}^{\gamma} = p_1 {v_1}^{\gamma}.$$

In other words, the pressure P of the gas sample is inversely proportional to the γth power of its volume V:

$$PV^{\gamma} = k \qquad \text{[Adiabatic Gas Law]},$$

<u>Example</u>. A sample of air has a volume of 182 cubic inches when it is under a pressure of 16.3 pounds per square inch. If the pressure is increased to 21.8 pounds per square inch under adiabatic conditions, what is the new volume?

<u>Solution</u>. If v is the resulting volume then

$$\begin{aligned}
21.8v^{1.4} &\doteq 16.3 \cdot 182^{1.4}.\\
v^{1.4} &\doteq \frac{16.3}{21.8} \cdot 182^{1.4}\\
v &\doteq 182 \cdot \left(\frac{16.3}{21.8}\right)^{1/1.4}\\
\log v &\doteq \log 182 + \frac{\log 16.3 - \log 21.8}{1.4}\\
&\doteq 2.2601 + \frac{9.8737 - 10}{1.4}\\
&= 2.2601 + \frac{13.8737 - 14}{1.4}\\
&\doteq 2.1699
\end{aligned}$$

So, the new volume is about 148 cubic inches.

EXERCISES

Solve these problems.

1. A sample of bromine gas [$\gamma \doteq 1.32$] occupies 140 cubic inches of volume when under a pressure of 23.4 pounds per square inch. If the pressure is reduced to 18.6 pounds per square inch, to what does the volume increase assuming adiabatic conditions?

2. Repeat Exercise 1 assuming isothermal conditions.

3. If you double the pressure on a sample of neon gas [$\gamma \doteq 1.64$] under adiabatic conditions, what change takes place in the volume?

4. Under adiabatic conditions, by what factor must you multiply the pressure of a sample of propane gas [$\gamma \doteq 1.13$] to reduce its volume by 50%?

5. Repeat Exercise 4 assuming isothermal conditions.

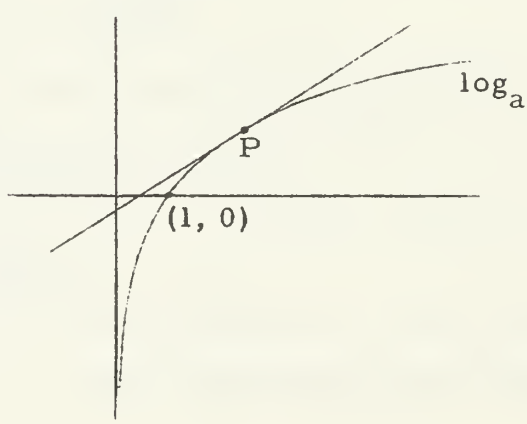
THE NUMBER e

In discussing radioactive decay we discovered the number e, one of whose definitions is:

$$e = \ell ub\{y: \exists_{x>0}\ y = (1 + x)^{1/x}\}$$

We shall now arrive at this number by a different approach.

Consider, for some $a > 1$, a graph of $\log_a$, and choose a point P on the graph. It is easy to guess that, among all the straight lines you



might draw through P, there is just one such that each point other than P on the graph of $\log_a$ lies below this line. This line is a graph of a linear function which is said to be the tangent to the function $\log_a$ at P. [What change needs to be made in this discussion if $0 < a < 1$?]

Now, let's focus our attention on the case in which P is the point $(1, 0)$. Let t_a be the linear function which is the tangent to $\log_a$ at $(1, 0)$, and let the number m_a be the slope of t_a. Then, by definition of a linear function, there is a number d such that

$$t_a = \{(x, y):\ y = m_a x + d\}.$$

Since $(1, 0)$ belongs to t_a, $d = -m_a$. Hence, for each x,

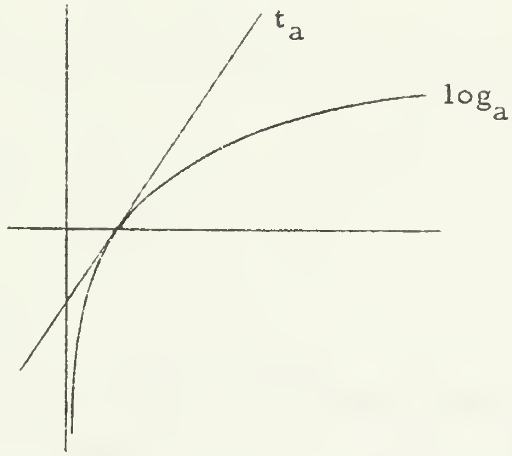
$$t_a(x) = m_a \cdot (x - 1).$$

The tangent t_a [for $a > 1$] is characterized by the fact that

(i) t_a is a linear function,

(ii) $t_a(1) = 0$,

and (iii) $\forall_{0 < x \neq 1}\ t_a(x) > \log_a(x)$.



Let's try, now, to compute m_a. We could, of course, get an approximation by making measurements on a graph. But, there is a better way.

Consider any other logarithm function for base greater than 1, say, $\log_b$. By our change-of-base formula [Theorem 219], for each $x > 0$,

$$\log_b x = \log_a x \cdot \log_b a.$$

Also, since $b > 1$ and $a > 1$, $\log_b a > 0$. Hence, since, for $0 < x \neq 1$,

$$t_a(x) > \log_a x \qquad \text{[see (iii), on p. 9-169]},$$

it follows that, for each such x,

(*) $$t_a(x) \cdot \log_b a > \log_a x \cdot \log_b a = \log_b x.$$

Now, since t_a is a linear function and $\log_b a$ is a constant, it follows that

$$t_a \cdot \log_b a \text{ is a linear function} \qquad \text{[with slope } m_a \cdot \log_b a\text{]}.$$

Also,

$$t_a(1) \cdot \log_b a = 0 \cdot \log_b a = 0.$$

So,

$$t_a \cdot \log_b a \text{ has the value 0 for the argument 1.}$$

Finally, by (*),

$$\forall_{0 < x \neq 1}\ t_a(x) \cdot \log_b a > \log_b x.$$

Thus, the function $t_a \cdot \log_b a$ meets the three conditions which characterize the tangent at (1, 0) to the function $\log_b$. So, $t_a \cdot \log_b a$ is the tangent t_b. Hence,

$$m_b = m_a \cdot \log_b a.$$

Now, for $a > 1$ and $b > 1$, it follows that

$$\left.\begin{aligned} b^{m_b} &= b^{m_a \cdot \log_b a} \\ &= \left(b^{\log_b a}\right)^{m_a} \end{aligned}\right\} \text{Theorem 207}$$

$$\left.\begin{aligned} \left(b^{\log_b a}\right)^{m_a} &= a^{m_a}. \end{aligned}\right\} \text{(L)}$$

In other words, the function f such that, for $a > 1$,

$$f(a) = a^{m_a}$$

is a constant function. It will turn out that the value of this constant function is the number e. [When we discover this, we shall know that $m_a = 1/\ell n\ a$. Explain.] However, since we do not yet know the value of the constant function, let's call it 'c'. So, for each $a > 1$,

$$a^{m_a} = c.$$

A glance at the graph of $\log_a$ [for $a > 1$] will convince us that, for $a > 1$, $m_a > 0$. So, since an exponential function whose base is greater than 1 is increasing, it follows that, for $a > 1$, $a^{m_a} > a^0 = 1$. Consequently, $c > 1$. So, c is an argument of f and--lifting ourselves by our boot-straps--

$$c^{m_c} = c.$$

Since the exponential function with base c has an inverse, it follows that

$$m_c = 1.$$

Now, m_c is the slope of the tangent at $(1, 0)$ to $\log_c$. So, for each x,

$$t_c(x) = m_c \cdot (x - 1) = x - 1.$$

Hence [see (iii) on page 9-169], for $0 < x \neq 1$,

$$\text{(1)} \qquad \log_c x < x - 1.$$

It is now not difficult to show that $c = e$. Keep in mind that, by definition,

$$e = \ell ub\{y: \exists_{x>0}\, y = (1 + x)^{1/x}\}.$$

We proceed as follows. Since, for $0 < x \neq 1$, $0 < \frac{1}{x} \neq 1$, it follows from (1) that, for each such x,

$$\log_c \frac{1}{x} < \frac{1}{x} - 1 = \frac{1 - x}{x}.$$

However, $\log_c \frac{1}{x} = -\log_c x$ and, so,

$$-\log_c x < \frac{1 - x}{x}$$

--that is, for $0 < x \neq 1$,

$$\text{(2)} \qquad \frac{x - 1}{x} < \log_c x.$$

Replacing 'x' in (1) and (2) by '$1 + x$', it follows that, for $-1 < x \neq 0$,

$$\text{(3)} \qquad \frac{x}{1 + x} < \log_c(1 + x) < x.$$

In particular [using Theorem 217], for $x > 0$,

$$\frac{1}{1 + x} < \log_c(1 + x)^{\frac{1}{x}} < 1.$$

Since $c > 1$, the exponential function with base c is increasing. So, it

follows that, for $x > 0$,

$$(4) \qquad c^{\frac{1}{1+x}} < (1 + x)^{\frac{1}{x}} < c.$$

From this result, it follows at once that c is an upper bound of

$$\{y : \exists_{x > 0}\, y = (1 + x)^{1/x}\}.$$

So, by definition $c \geq e$.

Suppose that $c > e$. In this case, $\log_e c > 1$, and there is a positive number x_0 such that $1 + x_0 = \log_e c$. It follows from Theorem 218b that

$$\frac{1}{1 + x_0} = \log_c e$$

and, so,

$$c^{\frac{1}{1+x_0}} = e.$$

Since, by definition,

$$e \geq (1 + x_0)^{\frac{1}{x_0}},$$

it follows that

$$c^{\frac{1}{1+x_0}} \geq (1 + x_0)^{\frac{1}{x_0}}.$$

But, by (4), this is not the case. Consequently, $c \not> e$ and, since $c \geq e$, $c = e$.

THE NATURAL LOGARITHM FUNCTION

We have seen that the function $\log_e$--for short, ℓn--is the logarithm function whose tangent at (1, 0) has slope 1. It is this property of ℓn which makes it, for many purposes, simpler than any other logarithm function and which accounts for its being called the natural logarithm function.

The simplicity of ℓn in comparison with the other logarithm functions can best be shown if we return to the considerations, on page 9-169, which led us to our re-discovery of the number e. As in that discussion, given any $x_1 > 0$, there is just one linear function--the tangent to ℓn at $(x_1, \ell n\, x_1)$--which contains the point $(x_1, \ell n\, x_1)$ and whose value, for each x such that $0 < x \neq x_1$, is greater than $\ell n\, x$. In particular, we have seen that when $x_1 = 1$, this linear function is the function t_e where, for each x,

$$t_e(x) = 1 \cdot (x - 1) = x - 1$$

--that is, we have seen that, for $0 < x \neq 1$,

$$(1) \qquad \ln x < x - 1.$$

It is now easy to find the tangent to $\ln$ at any point $(x_1, \ln x_1)$. For, using (1), for $0 < x/x_1 \neq 1$,

$$\ln \frac{x}{x_1} < \frac{x}{x_1} - 1$$

and, so, by Theorem 216, for $0 < x \neq x_1$,

$$(2) \qquad \ln x < \frac{x}{x_1} - 1 + \ln x_1.$$

Since the function t such that, for each x,

$$t(x) = \frac{x}{x_1} - 1 + \ln x_1$$

is a linear function which contains $(x_1, \ln x_1)$, it follows from (2) that t is the tangent to $\ln$ at $(x_1, \ln x_1)$. In particular, for each $x_1 > 0$, the slope of the tangent to $\ln$ at $(x_1, \ln x_1)$ is $1/x_1$.

EXERCISES

A. The tangent to $\log_a$ at $(x_1, \log_a x_1)$ was defined on page 9-169 [for $a > 1$ and $x_1 > 0$] to be the linear function t such that $t(x_1) = \log_a x_1$ and, for each positive $x \neq x_1$, $\log_a x < t(x)$. [In adopting this definition, we made the rather large assumption that, given $a > 1$ and $x_1 > 0$, there is one and only one such function t. This assumption can be justified, but the proof makes use of concepts which we shall not develop here.] In the case $a = e$--that is, for the function $\ln$--we proved, above, that, for $x_1 > 0$, the tangent to $\ln$ at $(x_1, \ln x_1)$ is such that

$$(*) \qquad t(x) = \frac{1}{x_1}(x - x_1) + \ln x_1.$$

1. Use (*) and the definition to find the tangent to $\log_a$ at $(x_1, \log_a x_1)$ for $a > 1$ and $x_1 > 0$]. [Hint. See the discussion on the top half page 9-170.]

2. How should the descriptions of the tangent to $\log_a$ at $(x_1, \log_a x_1)$ be modified to fit the case $0 < a < 1$?

3. Does the formula you found for t in Exercise 1 [where $a > 1$] also work if $0 < a < 1$?

4. The tangent to $\log_a$ at $(x_1, \log_a x_1)$ is the linear function which contains $(x_1, \log_a x_1)$ and whose graph lies [except at this one point] above the graph of $\log_a$ if $a > 1$, and below the graph of $\log_a$ if $0 < a < 1$. Find a single "directional phrase" [like, but different from, 'above' and 'below'] which can be used to describe the tangent in both cases.

5. For $0 < a \neq 1$, $\exp_a$ [that is, the exponential function with base a] is the inverse of $\log_a$. Guess a description of the tangent to $\exp_a$ at $(x_1, \exp_a x_1)$.

6. By (*) [and the definition of the tangent to ℓn at (1, 0)], for $0 < x \neq 1$,

$$\ell n\ x < x - 1.$$

From this infer that

$$\forall_{x \neq 0}\ e^x > 1 + x.$$

7. Use the result obtained in Exercise 6 to prove that, for $0 < a \neq 1$,

$$\forall_{x \neq 0}\ \exp_a(x) > 1 + x\ell n\ a.$$

8. Use the results of Exercises 5 and 7 to show that the tangent to $\exp_a$ at (1, 0) is the function t, where, for each x,

$$t(x) = 1 + x\ell n\ a.$$

9. Use the result of Exercise 6 to prove:

$$\forall_{x_1} \forall_{x \neq x_1}\ e^x > e^{x_1} + e^{x_1}(x - x_1)$$

[Hint. Consider using Theorem 206.]

10. From Exercises 5 and 9 argue that the slope of exp [that is, of the exponential function with base e] at $(x_1, \exp x_1)$ is $\exp x_1$.

B. Consider the function f where $f(x) = \frac{\ln x}{x}$, for $x > 0$.

1. Sketch what you think a graph of f might look like. [One ordered pair you should plot is $(e, f(e))$.]

2. We know that, for $x_1 > 0$ and for $0 < x \neq x_1$,

$$\ln x < \frac{1}{x_1}(x - x_1) + \ln x_1$$

and, so,

$$(*) \qquad f(x) < \frac{1}{x_1} - \frac{1}{x} + \frac{x_1}{x} f(x_1).$$

Check (*). Then, use it to show that, for $0 < x \neq e$, $f(x) < 1/e$. [This result, together with the fact that $1/e = f(e)$, shows that f takes on its maximum value at e. Does you graph show this?]

3. Show that f is decreasing on $\{x: x \geq e\}$. [Hint. From the result of Exercise 2 it follows that all you need to show is that if $e < x_1 < x_2$ then $f(x_2) < f(x_1)$. To do so, suppose that $e < x_1 < x_2$ and that $f(x_2) \geq f(x_1)$. Use this assumption together with (*) [with 'x_2' for 'x'] to infer that $f(x_1) < 1/x_1$. Use (*) again [with 'e' for 'x'] to obtain a contradiction.]

4. Show that f is increasing on $\{x: 0 < x \leq e\}$.

5. Show that, for all sufficiently large arguments, f has arbitrarily small positive values. [Hint. Use (*) to show that, given any positive number x_1, $f(x) < 2/x_1$ if x is sufficiently large.]

6. Assuming [as is the case] that f is continuous, your work in Exercises 3 and 5 shows that the range of the decreasing part of f is $\{y: 0 < y \leq 1/e\}$. Explain.

7. What is the range of the increasing part of f?

8. If you have not done so, modify the sketch you made in answer to Exercise 1 in order to show the information you now have about f.

C. 1. Which is larger, π^e or e^π?
[Hint. $\forall_{x>0} \forall_{y>0} [x^y > y^x \iff f(x) > f(y)]$].

2. Complete.

(a) For $e \leq y < x_1$, x^y ____ y^x.

(b) For $0 < y < x \leq e$, x^y ____ y^x.

(c) For $1 = y < x$, x^y ____ y^x.

3. Arrange in order of magnitude.

$$\sqrt{2}^{\sqrt{2}}, \quad 1.4^{\sqrt{2}}, \quad 1.4^{1.4}, \quad \sqrt{2}^{1.4}$$

4. Represent the results of Exercise 2 graphically. [Suggestion. Mark with '+'s the parts of a graph of $\{(x, y): 0 < y < x\}$ for which, by Exercise 2, $x^y > y^x$ and with '-'s those for which, by Exercise 2, $x^y < y^x$.]

5. Of course, $x^y = y^x$ if $x = y \geq 0$. But there are also points (x, y) with $0 < y < x$ for which $x^y = y^x$. Study the graph of f which you made in answering Exercise 8 of Part B and, on your answer for Exercise 4 of Part C, sketch $\{(x, y), 0 < y < x: x^y = y^x\}$.

6. Complete the graph you started in Exercise 4 to show, for $x > 0$ and $y > 0$, the points where $x^y > y^x$, $x^y = y^x$, and $x^y < y^x$.

D. 1. Note that, for each $x > 0$, $f(1/x) = -x\ln x$. Use this and your graph of f to sketch a graph of the function g, where $g(x) = x\ln x$. Where does g assume its minimum value?

2. The function $\exp \circ g$ [exp is the exponential function with base e] is the function h such that, for each $x > 0$, $h(x) = x^x$. Study your graph of g and use your knowledge of exp to sketch a graph of h.

3. Sketch a graph of $\exp \circ f$.

SUMMARY

In this unit you have learned more about logic [definite description], completed your set of basic principles for real numbers [the least upper bound principle], discovered and learned to use two important properties which some functions have [continuity and monotonicity], defined and studied three important kinds of functions [power functions, exponential functions, and logarithm functions], and learned some ways in which functions of these kinds "occur in nature". You also learned more about rational numbers [and, if you studied Appendix B, about irrational numbers and infinite sets], about mensuration formulas [if you studied Appendix D], about using logarithms to simplify computations, and [if you studied Appendix E] about functional equations. [Some of this last you learned, anyway, in seeing how laws of nature may be discovered.] Finally, you learned something about an important number--the number e.

In studying definite descriptions you learned that in using a phrase of this kind--say, 'the so-and-so'--you imply that there <u>is</u> a so-and-so and that there are not two so-and-sos. Hence--you learned--before using such a description, it is the better part of valor to make sure that the implied existence and uniqueness conditions are satisfied [or, at least, are not in contradiction with your previous commitments]. Once you have done this, it is then quite safe to use the definite description in question, and you can formalize this use by adopting a defining principle. For example, once you have proved:

$$(*_1) \qquad \forall_x \exists_z\ z^3 = x$$

and:

$$(*_2) \quad \forall_x \forall_y \forall_z\ [(y^3 = x \text{ and } z^3 = x) \Rightarrow y = z],$$

you can safely adopt the defining principle:

$$\forall_x\ (\sqrt[3]{x})^3 = x$$

Incidentally, from $(*_2)$ and the defining principles you can infer the theorem:

$$\forall_x \forall_y\ [y^3 = x \Rightarrow y = \sqrt[3]{x}]$$

Many of the applications you made of definite descriptions were, as in the preceding example, for the purpose of introducing inverses of functions with which you were already acquainted. [Many, but not all.

For example, you saw that the absolute value function and the greatest integer function--which, since they don't have inverses, are not inverses of other functions--could most simply be introduced by defining principles which arise out of the possibility of naming each of these functions by means of a definite description. You will presently be reminded--if you haven't already been--of another important example of this kind.]

In cases involving the inverse of a known function, satisfaction of the relevant uniqueness condition is established by proving that the known function has an inverse. The search for a standard way of proving such theorems led to the concept of monotonicity and to the basic result that monotonicity implies the existence of an inverse. The attempt to guarantee existence led to the adoption of the least upper bound principle; and the search for a standard way to prove the relevant existence theorems led to the concept of continuity. [The ℓubp is an existence principle which asserts that the "least upper bound function" whose arguments are sets of real numbers and whose values are real numbers, has for its domain the set of all nonempty sets of real numbers which have upper bounds. The corresponding uniqueness principle follows from the theorem according to which no set has two least numbers. Does the "least upper bound function" have an inverse?]

The general considerations which have just been summarized formed the basis for introducing and studying power functions, exponential functions, and logarithm functions. In Unit 8 we had already defined the integral power functions--

$$\{(x, y):\ y = x^k\},\quad \text{for some integer } k$$

[Sketch those for $k = 0,\ 1,\ 2,\ 3,\ -1$ and -2.]

--and the exponential sequences--

$$\{(k, y):\ y = a^k\},\ \text{for } a \neq 0, \text{ and } \{(k, y),\ k \geq 0:\ y = 0^k\}$$

[Sketch those for $a = 2,\ 1/2,\ -2$, and 0.]. The fact that each positive-integral power function is continuous and is increasing on the set of nonnegative numbers [and that those with odd exponents are increasing on the set of all real numbers] justified the introduction of the [continuous and increasing] principal root functions--

$$\{(x, y):\ y = \sqrt[2n-1]{x}\} \text{ and } \{(x, y),\ x \geq 0:\ y = \sqrt[2n]{x}\},\ \text{for some } n.$$

Using these root functions we were able to define the exponential functions--

$$\{(x, y): y = a^x\}, \text{ for } a > 0, \text{ and } \{(x, y), x \geq 0: y = 0^x\}.$$

Those with base $a > 0$ are continuous and, of these, those with base $a \neq 1$ are monotonic. Their inverses [which are, of course, continuous and monotonic] are the logarithm functions--

$$\{(x, y), x > 0: y = \log_b x\}, \ 0 < b \neq 1.$$

It turned out that the exponential and logarithm functions for a certain base--the number e--are particularly important.

$$e = \ell ub \{y: \exists_{x>0}\ y = (1 + x)^{1/x}\} \doteq 2.71828$$

The exponential function with base e is <u>the</u> exponential function, and is often denoted by 'exp'. The logarithm function to the base e is the <u>natural</u> logarithm function, and is often denoted by 'ℓn'. The function ℓn is distinguished from other logarithm functions by the fact that its slope at (1, 0) is 1 and, so, at $(x_0, \ell n\, x_0)$ is $1/x_0$, for $x_0 > 0$].

In terms of the exponential function we can define the power functions with positive arguments--

$$\{(x, y), x > 0: y = x^u\}, \text{ for some } u.$$

These functions are continuous and monotonic. When u is an integer, the power function with positive arguments whose exponent is u is a restriction of the corresponding integral power function of Unit 8. When u is the reciprocal of a positive integer n, the power function is a restriction of the principal nth root function. In general, for $u \neq 0$, the power functions with exponents u and $1/u$ are inverses of one another. [Using Theorem 217 and the defining principle (L) it is easy to see that the power function with positive arguments whose exponent is u is

$$\{(x, y), x > 0: y = e^{u \ell n\, x}\}.$$

This is often used as a definition of the power functions.]

Clearly, the exponential functions, the logarithm functions, and the power functions with positive arguments are closely related. If we consider the set of ordered triples

$$\{(x, y, z), x > 0: y = x^z\},$$

we see that the exponential function with base $a > 0$ corresponds to the "slice" of this set consisting of all its members for which $x = a$, and that the power function with exponent u and positive arguments corresponds to the slice consisting of all members with $z = u$. Also, using Theorems 217 and 219, the slice consisting of all members with $y = b$, for $0 < b \neq 1$, corresponds to the reciprocal of the logarithm function to the base b.

The basic nature of the exponential, logarithm, and power functions is brought out strikingly in Appendix E. There, as corollaries of more general results, it is shown that functions of these three kinds are the only monotonic functions which satisfy:

$$(2) \qquad \forall_x \forall_y\, f(x + y) = f(x)f(y),$$

$$(3) \qquad \forall_{x>0} \forall_{y>0}\, f(xy) = f(x) + f(y),$$

and:

$$(4) \qquad \forall_{x>0} \forall_{y>0}\, f(xy) = f(x)f(y),$$

respectively. The proofs given in Appendix E for these results depend on another one of a similar nature:

> The only monotonic functions which satisfy:
>
> $$(1) \qquad \forall_x \forall_y\, f(x + y) = f(x) + f(y)$$
>
> are the homogeneous linear functions--
>
> $$\{(x, y)\colon\ y = cx\}, \text{ for some } c \neq 0.$$

REVIEW EXERCISES

1. Critizize the following "defining principle" for square roots:

   For each $x \geq 0$,

   $\sqrt{x}$ is the number z such that $z^2 = x$.

2. Criticize the following "defining principle" for function composition:

   For all functions f and g,

   $f \circ g$ is the function h such that
   $h(x) = f(g(x))$, for each $x \in \mathcal{D}_g$.

3. Simplify.

   (a) $\sqrt{x^6}$ (b) $\sqrt{a^2 - 2ab + b^2}$ (c) $\sqrt{3x^2} + \sqrt{12x^2}$

   (d) $\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{4}}$ (e) $(2\sqrt{3} + 3\sqrt{2})^2 - (2\sqrt{3} - 3\sqrt{2})^2$

   (f) $\sqrt{\dfrac{3a^3bc^6}{4d^4e^2}}$ (g) $\dfrac{5 - \sqrt{3}}{2 + \sqrt{3}}$ (h) $\dfrac{a}{\sqrt{b}} + \dfrac{\sqrt{a}}{b}$

   (i) $\dfrac{2}{\sqrt{3} - 1}$ (j) $\dfrac{1}{1 - \sqrt{5} + \sqrt{7}}$ (k) $\sqrt{\dfrac{2x^{-2}y^3}{x^4y^{-3}}}$

4. Which is larger, $\sqrt{7} + \sqrt{10}$ or $\sqrt{3} + \sqrt{17}$?

5. Find the least upper bound of $\{x:\ 3(x - 4) < x\}$.

6. Find the greatest and least members of $\{x:\ |3x - 6| \leq 12\}$.

7. Find the least member of $\{x:\ 3 < x^2\}$.

8. Which of these functions are monotonic?

   (a) $\{(x, y):\ y = x^2\}$ (b) $\{(x, y),\ x \geq 4:\ y = x^2\}$

   (c) $\{(x, y):\ y = [\![x]\!]\}$ (d) $\{(x, y),\ x > 0:\ yx = 1\}$

9. Consider the function f where

$$f(x) = x - \frac{1}{x}, \quad \text{for } x > 0.$$

(a) Draw a graph of f.

(b) Prove that f is monotonic.

(c) Draw a graph of the inverse of f.

(d) You have probably guessed that the domain of the inverse of f is the set of all real numbers. Prove that it is. [Hint. One way is to argue from the continuity of f--you may assume that f is continuous--using Theorem 187. An easier way is to find a formula for computing values of the inverse.]

(e) You can introduce an operator--say, 'Q'--such that, for each x, Qx is the value of the inverse of f at x. For example, since $f(1) = 0$, $Q0 = 1$; and $Q{-4} = \sqrt{5} - 2$. State the appropriate defining principle:

$$\forall_x \, (Qx > 0 \quad \text{and} \qquad\qquad)$$

(f) From Exercise (b) it follows that f has an inverse and, so, that

$$\forall_{y>0} \forall_{z>0} \left[y - \frac{1}{y} = z - \frac{1}{z} \Rightarrow y = z\right].$$

From this and the defining principle of part (e), deduce the uniqueness theorem:

$$\forall_x \forall_{y>0} \left[y - \frac{1}{y} = x \Rightarrow y = Qx\right]$$

(g) Use the defining principle and the uniqueness theorem to prove:

$$\forall_x \, Q{-x} = 1/Qx$$

10. Suppose that f and g are monotonic functions such that there are m numbers x for which $f(x) = g(x)$. How many numbers x are there such that $f^{-1}(x) = g^{-1}(x)$?

11. Which of these functions are continuous?

(a) $f(x) = \begin{cases} x/3, & x \leq 6 \\ x/2, & 6 < x \end{cases}$

(b) $g(x) = \begin{cases} x/3, & x \leq 6 \\ 8 - x, & 6 < x \end{cases}$

12. Expand.

(a) $(\sqrt[3]{2} + 2)^5$ (b) $(\sqrt[4]{3} - 1)^4$

13. (a) Which three of the numbers $5 - \sqrt{3}$, $5 + \sqrt{3}$, $3 - \sqrt{2}$, and $-3 + \sqrt{2}$ are roots of the equation:

$$x^4 - 16x^3 + 89x^2 - 202x + 154 = 0?$$

(b) Find another root of this equation.

14. Simplify.

(a) $\sqrt[4]{81a^5b^8c^{17}}$ (b) $\sqrt[3]{-64a^6b^8}$ (c) $\sqrt[5]{64(x - y)^7}$

15. Show that the system of rational numbers is closed with respect to oppositing.

16. Which of these are irrational numbers?

(a) $\sqrt[3]{64}$ (b) $\sqrt[3]{28}$ (c) $\sqrt{5} + \sqrt{7}$ (d) $\dfrac{\sqrt{28}}{\sqrt{7}}$

17. Simplify.

(a) $49^{1/2}$ (b) $0.49^{0.5}$ (c) $0.0081^{-0.5}$ (d) $\left(3^{-12}\right)^{1/4}$

(e) $\left(\dfrac{3}{2}\right)^{-3}$ (f) $\left(\dfrac{16}{25}\right)^{-1/2}$ (g) $\left(\dfrac{27}{125}\right)^{-1/3}$ (h) $\left(-\dfrac{32}{y^{10}}\right)^{-1/5}$

(i) $\left(xy^{\frac{2}{3}}z^{\frac{1}{4}}\right)\left(x^{\frac{5}{6}}y^{\frac{3}{4}}z^{\frac{1}{2}}\right)\left(x^{\frac{3}{2}}y^{\frac{1}{6}}z^{\frac{2}{3}}\right)$

(j) $x^{-\frac{1}{3}}y^{-\frac{1}{3}}\left(x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}}\right)$

(k) $\left(x^{\frac{1}{3}} - 2y^{\frac{2}{3}}\right)\left(x^{\frac{1}{3}} + 2x^{\frac{1}{3}}y^{\frac{2}{3}} + 4y^{\frac{4}{3}}\right)$

18. Use the fact that, for all $x \geq 0$ and $y \geq 0$, $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$, to find the square roots of $23 + 4\sqrt{15}$.

19. Suppose that $g(x) = x^5 - x^4 - 7x^3 + 7x^2 + 10x$. Show that $g\left(2^{\frac{1}{2}}\right) = g\left(5^{\frac{1}{2}}\right)$.

20. Which of these numbers is smaller than $\sqrt{2}^{\sqrt{2}}$?

(a) $1.4^{\sqrt{2}}$ (b) $\sqrt{3}^{\sqrt{2}}$ (c) $\sqrt{2}^{1.4}$ (d) $\sqrt{2}^{\sqrt{3}}$ (e) $1.4^{1.4}$

21. Show that $\log_c b \cdot \log_a c \cdot \log_b a = 1$.

22. Use logarithms to compute an approximation to $\sqrt{(28.39)^2 - (12.71)^2}$.

23. Compute the length of a radius of a solid sphere whose volume is 726.3 cubic inches.

24. Solve these equations.

(a) $5^{2x-3} = 10$ (b) $9^{x^2} = 100$ (c) $2^x = 3^{x-1}$

(d) $\log(x+1) - \log x = \log 3$ (e) $\log x - \log(x+1) = 2$

25. Suppose that f is log and g is $\log_5$. Complete each of the following sentences.

(a) $\forall_{x>0}\ g(x) = \dfrac{f(x)}{\qquad} = \qquad \cdot g(\quad)$

(b) $\forall_{x>0}\ [f(x) = g(x) \Longleftrightarrow x = \qquad]$

(c) $\forall_{x>0}\ f(x^2) = 2 \cdot$

(d) $\forall_{y>0}\ g(\sqrt{y}) =$

(e) $\forall_{x>0} \forall_{y>0}\ f(xy) = \qquad +$

26. Suppose that f is the exponential function with base 3 and g is the linear function with slope 6 and intercept −3. Find, by inspection, all numbers x such that $f(x) \leq g(x)$. [Hint. Sketch the graphs of f and g.]

27. Given the functions g and f where, for each x, $g(x) = 2^x$ and $f(x) = x^2$. How many numbers x are there such that $f(x) = g(x)$?

28. In a room in which the temperature is 20°C a piece of iron cools from 350°C to 75°C in 15 minutes. Assuming that Newton's Law of Cooling applies [page 9-158], find how long it took the metal to cool from 350° to 100°C.

29. The intensity of a light as seen through fog diminishes with distance according to the formula:

$$I(d) = I(0)e^{-kd}$$

Suppose that the intensity diminishes by 50% at 15 feet from the source.

(a) Compute k.

(b) How far from the source can the light be seen by a man who is capable of perceiving a light whose intensity is one ten thousandth that of the source?

30. To say that a function has an inverse is to say that it matches the members of its domain with the members of its range in a one-to-one manner--that is, it determines a one-to-one correspondence between the members of its domain and the members of its range.

<u>Examples</u>. The function g where $g(n) = n + 1$, for all positive integers n, determines a one-to-one correspondence between the positive integers and the positive integers greater than 1.

Any exponential function with positive base different from 1 determines a one-to-one correspondence between the real numbers and the positive numbers.

(a) Consider the function f, where

$$f(x) = \frac{x}{1 + x}, \qquad \text{for } x > 0.$$

Does f have an inverse? What is the range of f?

(b) Your answers to part (a) tell you that f determines a one-to-one correspondence between all positive numbers and the members of some set of real numbers. What set?

(c) Use what you have discovered in (b) and the fact that the exponential function $[y = e^x]$ determines a one-to-one correspondence between the real numbers and the positive numbers to find

a function which determines a one-to-one correspondence between all real numbers and all numbers between 0 and 1.

(d) Show that the function g, where

$$g(x) = \frac{x}{1 + |x|},$$

determines a one-to-one correspondence between all real numbers and those between -1 and 1. [Note: $\forall_x\, g(-x) = -g(x)$.]

(e) Starting with the function g of part (d), find a constant function c such that g + c determines a one-to-one correspondence between all real numbers and all numbers between 3 and 5.

(f) Starting with the function g of part (d), define a function h which determines a one-to-one correspondence between all real numbers and all numbers between 0 and 1.

(g) Find a formula for computing values of the inverse of g. [Hint. You need to solve the equation:

$$x = \frac{y}{1 + |y|}$$

for 'y'. Consider two cases: $x \geq 0$ and: $x < 0$]

MISCELLANEOUS EXERCISES

1. Factor.

(a) $(3x + 1)^2 - (2x - 1)^2$

(b) $x^2 - 4xy + 4y^2 - 9x^2y^2$

(c) $r^2 + 8r^3$

(d) $8a^3 - b^3$

(e) $a^4 + 8a^2 + 7$

(f) $b^2 - 29ba + 54a^2$

(g) $t^4 - 16s^2$

(h) $25 - 64a^2$

(i) $t^2 + t - 56$

(j) $m^2 - m - 20$

2. Solve.

(a) $\dfrac{7x^2 + 8}{21} = \dfrac{x^2}{3} + \dfrac{x^2 + 4}{8x^2 - 11}$

(b) $\dfrac{x + 3}{7 - 2x} = \dfrac{1 - 2x}{x - 3}$

3. Suppose that a is a geometric progression. Show that

$$\forall_n \forall_p \forall_q [(p \leq n \text{ and } q \leq n) \Rightarrow a_{n+p}a_{n-p} = a_{n+q}a_{n-q}].$$

4. Solve: $\begin{cases} 2^{x+4} = 6 \\ 2^{x+1} = 3^y \end{cases}$ 5. Solve: $x^{\frac{x}{3}} = 36$ 6. Solve: $\log(2x+4) = 2$

7. Solve: $x\sqrt[x]{x} = x^x$ 8. Solve: $\log x^6 = 12$ 9. Solve: $4^{2x-3} = 2^{2x+6}$

10. Show that if a, b, and c are consecutive terms of an AP then so are $b^2 + bc + c^2$, $c^2 + ac + a^2$, and $a^2 + ab + b^2$. Is the converse true?

11. **Show that if each term of a nonconstant geometric progression is subtracted from the next term, the successive differences are the terms of a geometric progression.**

12. If $A^2 + B^2 = 7AB$, show that $\log\frac{1}{3}(A + B) = \frac{1}{2}(\log A + \log B)$.

13. Solve for 'x': $(a^4 - 2a^2b^2 + b^4)^{x-1} = (a - b)^{2x}(a + b)^{-2}$

14. Solve: $x^{x\sqrt{x}} = (x\sqrt{x})^x$

15. Factor.

(a) $1000x^2y - 40y^3$ (b) $x^5 - 8x^2y^3$

(c) $5a^4 - 15a^3 - 90a^2$ (d) $1 - (x^2 + y^2) + 2xy$

(e) $\frac{8}{x^3} - 27y^6$ (f) $\frac{27}{x^3y^3} - 1$

(g) $(a + 5b)^2 - 4b^2$ (h) $(2a - 3b)^2 - 9b^2$

(i) $16a^2 - (2b - 3c)^2$ (j) $(m + n)^2 - (p + q)^2$

16. Complete: $\forall_n \sum_{p=1}^{2n} 2^p =$

17. Solve these equations.

(a) $\frac{3(k - 1)}{16} - \frac{5}{12}(k - 4) = \frac{2}{5}(k - 6) + \frac{5}{48}$

(b) $\frac{3t}{4} - \frac{6}{17}(t + 10) - (t - 3) = \frac{t - 7}{51} - \frac{19}{4}$

18. Find the arithmetic progression a with common difference 7 such that

$$\forall_m \forall_n \; n^2 \sum_{p=1}^{m} a_p = m^2 \sum_{p=1}^{n} a_p .$$

19. Expand, and express without referring to negative exponents.

(a) $(x + 5 + 6x^{-1})(1 + 6x^{-1} + 8x^{-2})$

(b) $(3x - 8)[x - 1 - (1 - x[4 + x]^{-1})^{-1}]^{-1}$

20. Simplify.

(a) $a + b + \frac{a^2 - ab}{b}$

(b) $x^3 + xy^2 + \frac{xy^4}{x^2 - y^2}$

21. Simplify.

(a) $\frac{a^2 + b^2}{ab} - \frac{a^2}{ab + b^2} - \frac{b^2}{a^2 + ab}$

(b) $\frac{y - x}{x + y} + \frac{2y^2 - 2xy}{x^2 - y^2}$

22. Simplify.

(a) $17.4 \times 8.6 - 7.4 \times 8.6$

(b) $18.8 \times 12.4 - 17.6 \times 6.2$

(c) $24 \cdot 13 + 9 \cdot 26 - 4 \cdot 39$

(d) $12 \cdot 14 + 4 \cdot 28 - 5 \cdot 28$

(e) $\frac{7 \times 23 - 14 \times 5}{13 \times 18 + 26 \times 12}$

(f) $\frac{39 \times 7 + 21 \times 21}{9 \times 34 - 12 \times 17}$

23. If n arithmetic means are inserted between 1 and n^2, what is the smallest of these means?

24. If the square of A varies as the cube of B, and A has the value 3 when B has the value 4, what value of B corresponds with the value $\sqrt{3}/3$ of A?

25. A certain number of persons shared equally in the cost of a party. If there had been 10 more, each would have paid \$1 less; if there had been 5 fewer, each would have paid \$1 more. How many persons shared the cost?

26. Two numbers are in the ratio 7:12. Find the smaller if the larger exceeds it by 275.

27. Solve for the indicated variable.

(a) $t = 2\pi\sqrt{\dfrac{\ell a^3}{8T}}$; T (b) $S = \dfrac{76v}{12v + 10.6m}$; v

28. A wagon wheel of radius r picks up a piece of paper from the road and carries it round 1/12 of a revolution. How high is it above the ground? How high will it be when it has gone 7/12 of the way round?

29. Simplify.

(a) $\dfrac{x + y}{2x + z} - \dfrac{2yz}{2x^2 + 5xz + 2z^2}$ (b) $\dfrac{ab}{9a^2 - 16b^2} + \dfrac{b}{6a - 8b}$

30. If the ratio of x to y is 3 to 4, what is the ratio of $7x - 4y$ to $3x + y$?

31. Find a root of '$27x^{1.5} = 1$'.

32. Express 93.105003 as an indicated sum of multiples of powers of 10.

33. Solve the system: $\begin{cases} 4 \cdot 3^{2 - x - y} = 3 \cdot 4^{1 - y} \\ 3 \cdot 2^{2x} = 2 \cdot 3^{3y - x + 1} \end{cases}$

34. Solve: $\left(\dfrac{3}{4}\right)^{\log x} + \left(\dfrac{4}{3}\right)^{\log x} = \dfrac{25}{12}$

35. How many ordered pairs of integers (k, j) are there such that $0 \leq j \leq 2^k$ and $k \leq 5$?

36. Complete:

(a) $\displaystyle\sum_{p=1}^{100} [1 + (-1)^p] =$ (b) $\displaystyle\sum_{p=1}^{99} [1 + (-1)^p] =$

(c) $\forall_n \displaystyle\sum_{p=1}^{2n} [1 + (-1)^p] =$ (d) $\forall_n \displaystyle\sum_{p=1}^{n} [1 + (-1)^p] =$

APPENDIX A

[This Appendix deals in a more detailed manner with the material covered in section 9.04 and the Exploration Exercises which precede that section. In particular, it contains proofs of the fundamental Theorems 184 and 187 on the positive-integral power functions.]

The simplest functions.--We have seen in section 9.01 that the justification of our work with the principal square root operator lies in two theorems about the squaring function:

$$(\dagger_1)\ \forall_{x\geq 0}\ \exists_z\ (z\geq 0 \text{ and } z^2 = x)$$

and:

$$(\dagger_2)\ \forall_{x\geq 0}\ \forall_y\ \forall_z[((y\geq 0 \text{ and } y^2 = x) \text{ and } (z\geq 0 \text{ and } z^2 = x)) \Rightarrow y = x]$$

The first of these can be abbreviated to:

$$(1)\ \forall_{y\geq 0}\ \exists_{x\geq 0}\ x^2 = y \qquad [\text{'y' for 'x'; 'x' for 'z'}]$$

and the second is equivalent to:

$$(2)\ \forall_{x_1\geq 0}\ \forall_{x_2\geq 0}[x_1^2 = x_2^2 \Rightarrow x_1 = x_2] \quad [\text{'}x_1\text{' for 'y', '}x_2\text{' for 'z'}]$$

[Although (2) is simpler than $(\dagger_2)$ it is not hard to see that, in view of the fact that squares of nonnegative numbers are nonnegative, (2) and $(\dagger_2)$ are equivalent. For, in the first place, because of the fact just cited, the restriction '$x\geq 0$' in $(\dagger_2)$ is unnecessary [according to the antecedent of $(\dagger_2)$, x is the square of the nonnegative number y and, so, is bound to be nonnegative]. And, in the second place, the 'x's in the antecedent of $(\dagger_2)$ serve merely to guarantee '$y^2 = z^2$'. So, in view of the fact about squares, $(\dagger_2)$ is equivalent to:

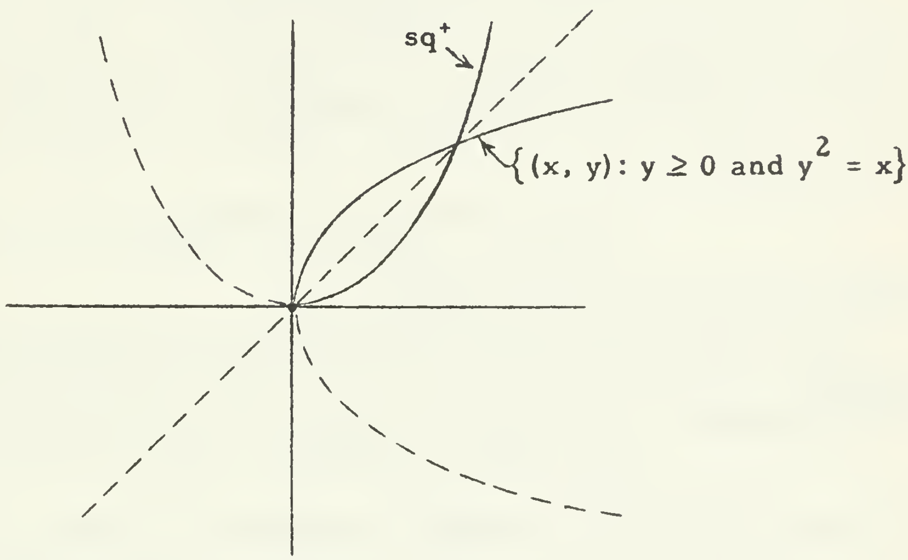
$$\forall_y\ \forall_z\ [(y\geq 0 \text{ and } z\geq 0 \text{ and } y^2 = z^2) \Rightarrow y = z]$$

And, (2) is merely an abbreviation of this.]

We shall develop a better insight into how an operator like '$\sqrt{\ }$' [for example, '$\sqrt[3]{\ }$', '$\sqrt[18]{\ }$', and other operators which you have not yet heard of] can be justified by concentrating on a related function [for '$\sqrt{\ }$', the squaring function; for the examples mentioned, the cubing function, the 18th power function, and, for one you probably haven't heard of, the exponential function with base 2].

Looking at (1) and (2), we see $[x \geq 0,\ x_1 \geq 0,\ x_2 \geq 0]$ that these are really statements about a subset of the squaring function--namely, the function sq^+ defined by:

$$sq^+ = \{(x,\ y),\ x \geq 0 : y = x^2\}$$



In terms of sq^+, statements (1) and (2) become:

$$(1')\quad \forall_{y \geq 0} \exists_{x \in \mathcal{D}_{sq^+}} sq^+(x) = y$$

and:

$$(2')\quad \forall_{x_1 \in \mathcal{D}_{sq^+}} \forall_{x_2 \in \mathcal{D}_{sq^+}} [sq^+(x_1) = sq^+(x_2) \Rightarrow x_1 = x_2]$$

In geometric terms, (1′) tells us that each horizontal line with nonnegative intercept $[y \geq 0]$ crosses the graph of sq^+--there are no holes in the graph. (2′) tells us that each horizontal line that crosses the graph does so in precisely one point.

If we recall that a function has an inverse if no two of its ordered pairs have the same second component [that is, if its converse is a function], we see that (2′) says merely that sq^+ has an inverse. As soon as we know [by (2′)] that sq^+ has an inverse, we are justified in speaking of the principal square root of any number in the range of sq^+. Doing so would not be of much use unless we knew what numbers belong to the range of sq^+, that is, to the domain of its inverse. (1′) tells us part of the story--each nonnegative number $[\forall_{y \geq 0} \ldots]$ belongs to the range of

sq^+. And, since we know that

$$\forall_{x \geq 0}\ sq^+(x) \geq 0, \qquad \text{[Theorems 15 and 97a]}$$

it follows that $\mathcal{R}_{sq^+}$ is $\{y: y \geq 0\}$.

Once we have proved (1′) and (2′), we know that sq^+ has an inverse whose domain is the set of nonnegative numbers. Consequently, we may then introduce an operator, '$\sqrt{\ }$', to use in referring to this inverse. We may, as we have seen in section 9.01, do so by adopting a defining principle:

$$\forall_{x \geq 0}\ (\sqrt{x} \in \mathcal{D}_{sq^+} \text{ and } sq^+(\sqrt{x}) = x)$$

[Compare this with $(*_1)$ on page 9 - 5.]

From this defining principle and (2′) we can deduce a uniqueness theorem:

$$\forall_{x \geq 0}\ \forall_{y \in \mathcal{D}_{sq^+}} [\, sq^+(y) = x \Rightarrow y = \sqrt{x}\,]$$

[Compare this with $(*_2)$ on page 9 - 5.]

Since sq^+ is a subset of the more familiar squaring function, we may reformulate the defining principle and the uniqueness theorem in more familiar terms by eliminating 'sq^+' in favor of the operator '^{2}'. Recalling that $\mathcal{D}_{sq^+} = \{x: x \geq 0\}$ we obtain the defining principle:

$$(*_1) \qquad \forall_{x \geq 0}\ (\sqrt{x} \geq 0 \text{ and } (\sqrt{x})^2 = x)$$

and the uniqueness theorem:

$$\forall_{x \geq 0}\ \forall_{y \geq 0} [\, y^2 = x \Rightarrow y = \sqrt{x}\,]$$

Since each number belongs to the domain of the squaring function, this uniqueness theorem is merely an abbreviation of:

$$(*_2) \qquad \forall_{x \geq 0}\ \forall_y [\, (y \geq 0 \text{ and } y^2 = x) \Rightarrow y = \sqrt{x}\,]$$

The preceding discussion illustrates a general procedure for justifying the introduction of an operator [like '$\sqrt{\ }$'] to abbreviate a definite description ['the real number z such that $z \geq 0$ and $z^2 =$ ']. Such a description refers to a function f_o [the function sq^+] which is a subset ['$z \geq 0$'] of a known function f [''z^2''--the squaring function]. The procedure is to show that this subset f_o, whose domain is some subset D of

$\mathfrak{D}_{f}$, has an inverse whose domain is a certain set R [$\{x: x \geq 0\}$]. This is done by proving, first, that R is contained in the range of f_o:

$$\text{(I)} \quad \forall_{y \in R} \exists_{x \in D}\, f(x) = y \qquad \text{[Compare with (1) on page 9-190.]}$$

and that the range of f_o is contained in R [$\forall_{x \geq 0}\ sq^+(x) \geq 0$]. [In most cases this second inclusion is already known--just as we know that all squares of real numbers are nonnegative, we shall usually know that the range of the entire function f is contained in R.] Secondly, one proves that f_o has an inverse:

$$\text{(II)} \quad \forall_{x_1 \in D}\ \forall_{x_2 \in D} [f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$$

[Compare with (2) on page 9-190.]

Having done these things, we are entitled to introduce a name--say, 'g' --[or an operator] for the inverse of f_o, and we do so by adopting a defining principle:

$$(\star_1) \quad \forall_{x \in R} (g(x) \in D \text{ and } f(g(x)) = x)$$

From $(\star_1)$ and (II) we then have the uniqueness theorem:

$$(\star_2) \quad \forall_{x \in R} \forall_{y \in D} [f(y) = x \Rightarrow y = g(x)]$$

[If the domain of the entire function f is the set of all real numbers, we may rewrite $(\star_2)$ as:

$$\forall_{x \in R}\ \forall_y [(y \in D \text{ and } f(y) = x) \Rightarrow y = g(x)] \qquad]$$

Now that we have seen that the problem of justifying the introduction of an operator often amounts to proving [(II)] that a subset of a related function has an inverse and to establishing [(I)] what the domain of this inverse is, we shall proceed to discover a large class of functions--the <u>monotonic functions</u>--each of which has an inverse. This will give us an easy way to prove the theorems we wish to of the form (II). Later in this Appendix we shall show how to determine the ranges of many of these monotonic functions--namely of those which are continuous and which [like sq^+] have "unbroken" domains [$\mathfrak{D}_{sq^+}$ is a ray; rays, segments, intervals, half-lines, and the set of all real numbers are examples of what we mean by 'unbroken domains']. This will give us an easy way to prove the theorems we wish to of the form (I).

MONOTONIC FUNCTIONS

To discover a property of functions which insures that they have inverses, let's return to the example of sq^+ and take note of one way of proving (2) on page 9-190:

$$\forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_1^2 = x_2^2 \Rightarrow x_1 = x_2]$$

[You may already have proved (2), probably in a different way, in Unit 7. For, as you probably have noticed, (2) is Theorem 98a.] Our proof depends heavily on an order theorem about the squaring function. By Theorem 98c, for $a \geq 0$,

$$\text{if } b > a \text{ then } b^2 > a^2$$

and, for $b \geq 0$,

$$\text{if } a > b \text{ then } a^2 > b^2.$$

Suppose, now, that $a \neq b$. It follows [from Theorem 86a] that

$$b > a \quad \text{or} \quad a > b.$$

For $a \geq 0$ and $b \geq 0$, it then follows that

$$b^2 > a^2 \quad \text{or} \quad a^2 > b^2.$$

In both cases [by Theorem 87], $a^2 \neq b^2$. Consequently [by contraposition], for $a \geq 0$ and $b \geq 0$,

$$\text{if } a^2 = b^2 \text{ then } a = b.$$

So, using the general properties of order formulated in Theorems 86a and 87, the fact that sq^+ has an inverse follows from Theorem 98c. This latter theorem is, in view of the definition of sq^+ [and transitivity], equivalent to:

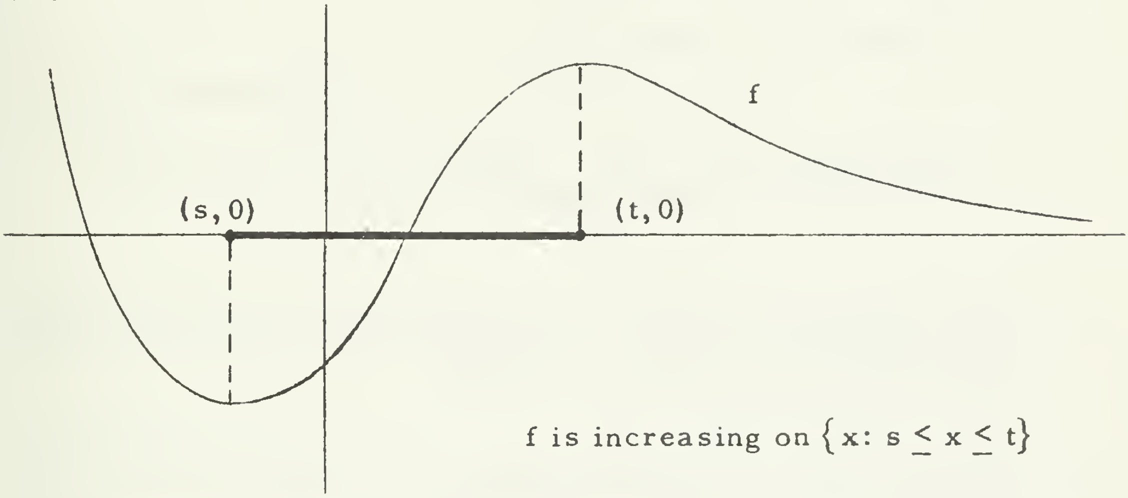
$$\forall_{x_1 \in \mathcal{D}_{sq^+}} \forall_{x_2 \in \mathcal{D}_{sq^+}} [x_2 > x_1 \Longrightarrow sq^+(x_2) > sq^+(x_1)]$$

For short, we say that the function sq^+ is an <u>increasing</u> function.

<u>Definition.</u> A function f is increasing on a set D if and only if $D \subseteq \mathcal{D}_f$ and $\forall_{x_1 \in D} \forall_{x_2 \in D} [x_2 > x_1 \Rightarrow f(x_2) > f(x_1)]$. A function which is increasing on its domain is called (merely) <u>an increasing function</u>.

Notice that an increasing function is increasing on each subset of its domain--in other words, each subset of an increasing function is an increasing function.

The same procedure we used above in proving Theorem 98a can be used to prove that each increasing function has an inverse. [See Part B on page 9-196.]



f is increasing on $\{x \colon s \leq x \leq t\}$

For the function f shown above, name a set on which f is decreasing. Name another.

EXERCISES

A. 1. Formulate a definition of a function decreasing on a set D:

A function f is decreasing on a set D

$\underline{\qquad ? \qquad}$

$D \subseteq \underline{\ ?\ }$ and $\forall_{x_1 \in D} \forall_{x_2 \in D} [x_2 > x_1 \Rightarrow \underline{\qquad ? \qquad}]$

2. Formulate a definition of a decreasing function.

*

A function which is either increasing on D or decreasing on D is said to be monotonic on D.

A function which is either an increasing function or a decreasing function is said to be $\underline{\qquad ? \qquad}$.

*

B. 1. Prove that each increasing function has an inverse.

Suppose that f is an increasing function. By definition, for $a \in \mathcal{D}_f$ and $b \in$ __?__ ,

if $b > a$ then $f(b) > f(a)$

and if $a > b$ then ___?___ .

Suppose, now, that $a \neq b$...

.
. ←[Complete the proof.]
.

Therefore, $\forall_{x_1 \in \mathcal{D}_f} \forall_{x_2 \in \mathcal{D}_f} [f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$.

2. How would you change the foregoing proof to a proof that each decreasing function has an inverse?

3. Suppose that f is an increasing function. By Exercise 1, we know that f has an inverse, say, g. What kind of function is g? [Make some sketches to test your discovery.]

4. Prove that if f is an increasing function then

$$\forall_{x_1 \in \mathcal{D}_f} \forall_{x_2 \in \mathcal{D}_f} [f(x_2) > f(x_1) \Rightarrow x_2 > x_1].$$

5. Prove that the inverse of each increasing function is also increasing. [Hint. Suppose that f is an increasing function, that g is the inverse of f, and, for c and d in $\mathcal{D}_g$, that $d > c$. Now, use the defining principle $(\star_1)$ on page 9-193 (with $R = \mathcal{D}_g$ and $D = \mathcal{D}_f$) and the theorem of Exercise 4.]

6. How would you modify your work in Exercises 4 and 5 to show that the inverse of a decreasing function is also a decreasing function?

* * *

The work you have done in Part B amounts to proving:

Theorem 184.
Each monotonic function has a monotonic inverse of the same type.

* * *

C. 1. (a) Use the definition of a monotonic function to show that the squaring function is not monotonic.

(b) Use the theorem that each monotonic function has an inverse to show that the squaring function is not monotonic.

2. Is the function sq^- monotonic? If so, what is its inverse?

3. Suppose that r is the reciprocating function--that is, that

$$r = \{(x, y), x \neq 0: y = x^{-1}\}.$$

(a) Sketch a graph of r.

(b) Does r have an inverse?

(c) Prove that r is not monotonic.

(d) Are there sets on which r is monotonic?

(e) Is the domain of r the union of two sets on each of which r is monotonic?

4. Repeat Exercise 3 for the functions r^2 and r^3.

5. The functions which you have considered in Exercises 3 and 4 are the first three negative-integral power functions. On the basis of your work in these exercises make general remarks on monotonicity and existence of inverses of the negative-integral power functions.

6. Investigate the positive-integral power functions. [The second of these is, of course, the squaring function.]

D. 1. Prove, again, that sq^+ is an increasing function. That is, prove:

$$\forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_2 > x_1 \Rightarrow x_2^2 > x_1^2]$$

Do this by using the algebra theorem:

$$\forall_{x_1} \forall_{x_2} \; x_2^2 - x_1^2 = (x_2 - x_1)x_2 + (x_2 - x_1)x_1$$

[Hint. You know that, for each $x > 0$ and $y > 0$, $xy > 0$ and $x + y > 0$. From this, together with the pm0 and the pa0, it is easy to see that

(*) $$\forall_{x>0} \forall_{y \geq 0} (xy \geq 0 \text{ and } x + y > 0).$$

The theorems just mentioned, together with the algebra theorem and Theorems 84 and 92 should suffice for your proof.]

2. Prove that the positive-integral power functions, restricted to nonnegative arguments, are increasing functions. That is, prove [by mathematical induction]:

$$\forall_n \forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_2 > x_1 \Rightarrow x_2^n > x_1^n] \qquad \text{[Theorem 185]}$$

Do this by using the algebra theorem:

$$\forall_n \forall_{x_1} \forall_{x_2} [x_2^{n+1} - x_1^{n+1} = (x_2 - x_1)x_2^n + (x_2^n - x_1^n)x_1]$$

[Hint. Your answer for Exercise 1 should suggest part (ii) of the inductive proof. In this part you will also need Theorem 152a.]

* * *

In the investigation you carried out in answering Exercise 6 of Part C, you probably discovered that the odd positive-integral power functions are increasing functions, but that the even positive-integral power functions are increasing on $\{x: x \geq 0\}$ and decreasing on $\{x: x \leq 0\}$. It is not hard to derive these results by using Theorem 185.

The difference between the behavior of the even power functions and that of the odd power functions is due to the fact that

(*) $$\forall_n \forall_x ((-x)^{2n} = x^{2n} \text{ and } (-x)^{2n-1} = -x^{2n-1}),$$

[(*) is an easy consequence of Theorems 28, 158, and 150c.] Here is a

proof, consisting of two parts, that each odd positive-integral power function is increasing:

Suppose, for $a_1 \le 0$ and $a_2 \le 0$, that $a_2 > a_1$. It follows that $-a_2 \ge 0$, $-a_1 \ge 0$, and $-a_1 > -a_2$. Hence, by Theorem 185, $(-a_1)^{2p-1} > (-a_2)^{2p-1}$. So, by (*), $-a_1^{2p-1} > -a_2^{2p-1}$ --that is [Theorem 94], $a_2^{2p-1} > a_1^{2p-1}$. Hence, for $a_1 \le 0$ and $a_2 \le 0$, if $a_2 > a_1$ then $a_2^{2p-1} > a_1^{2p-1}$. Consequently, each odd positive-integral power function is increasing on $\{x: x \le 0\}$.

Now, since each odd positive-integral power function is increasing on $\{x: x \le 0\}$ and, by Theorem 185, is also increasing on $\{x: x \ge 0\}$, it follows [Theorem 86c] that each such function is an increasing function [Explain. Hint. If $a_2 > a_1$ then, either a_2 and a_1 are both nonpositive or a_2 and a_1 are both nonnegative, or $a_2 > 0 > a_1$.]

So, we have proved:

$$\forall_n \forall_{x_1} \forall_{x_2} [x_2 > x_1 \Rightarrow x_2^{2n-1} > x_1^{2n-1}] \quad \text{[Theorem 185']}$$

* * *

☆E. 1. Repeat the first part of the proof of Theorem 185′ with '2p - 1' replaced by '2p' and, so, obtain a theorem about even positive-integral power functions restricted to nonpositive arguments.

2. If, for some p, the function g is the inverse of the 2pth power function restricted to nonnegative arguments, what is the inverse of the 2pth power function restricted to nonpositive arguments?

3. Recall [Exercise 3 of Part C] that the reciprocating function, r, is decreasing on $\{x: x < 0\}$ and on $\{x: x > 0\}$, but not on $\{x: x \neq 0\}$. But, why cannot the second part of the proof of Theorem 185′ be modified to prove that r is decreasing on $\{x: x \neq 0\}$?

* * *

On page 9-193 we reached the conclusion that the introduction of an operator can often be justified by proving that a certain function has an inverse and establishing what is the domain of this inverse--that is, what is the range of the function. So, it became our goal to discover generalizations from which, for each of many functions, it would be easy to deduce that the function in question has an inverse and easy to establish what the range of the function in question is.

Theorem 184 is one of the sought-for generalizations. It tells us that if a function f is monotonic on a set D then the function obtained by restricting f to the set D has an inverse. In other words, if f is monotonic on D then [see page 9-193]

$$\text{(II)} \qquad \forall_{x_1 \in D} \forall_{x_2 \in D} [f(x_1) = f(x_2) \Rightarrow x_1 = x_2].$$

For example, since [Theorem 185′] each odd positive-integral power function [f] is monotonic on the set of all real numbers, we know [D = the set of all real numbers] that

$$\text{(a)} \qquad \forall_n \forall_{x_1} \forall_{x_2} [x_1^{2n-1} = x_2^{2n-1} \Rightarrow x_1 = x_2].$$

Also, since [Theorem 185] each positive-integral power function [f] is monotonic on the set of nonnegative numbers, we know $[D = \{x: x \geq 0\}]$ that

$$\text{(b)} \qquad \forall_n \forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_1^n = x_2^n \Rightarrow x_1 = x_2].$$

A theorem of the form (II) justifies our introducing an operator which gives us, for each number in the range of the function obtained by restricting f to arguments in D, the value of the inverse of this function at this number. For example, the theorem (b) gives us the right to introduce an operator [it will be '$\sqrt[n]{}$'] which, for each n, gives us the values of the inverse of the restricted nth power function. However, before we can use this operator properly, we must find out for which numbers this inverse is defined--that is, we must determine the ranges of the restricted power functions. The major step in doing so consists in proving theorems of the form [see page 9-193]:

$$\text{(I)} \qquad \forall_{y \in R} \exists_{x \in D} f(x) = y$$

Using the notion of continuity, which you will discover in the Exploration

Exercises which follow, we shall be able to prove theorems of this kind. In particular, we shall be able to prove:

(c) $$\forall_n \forall_{y \geq 0} \exists_{x \geq 0} x^n = y$$

This, together with the rather easy theorem:

$$\forall_n \forall_{x \geq 0} x^n \geq 0,$$

shows that the range of each of the restricted positive-integral power functions is $\{y: y \geq 0\}$. So, the operator '$\sqrt[n]{\ }$' can [like '$\sqrt{\ }$'] be applied to each nonnegative number--for each n, and for each $x \geq 0$, we are, by the theorems (b) and (c), entitled to speak of <u>the</u> principal nth root of x. [Similarly, the theorem (a), together with another theorem like (c):

$$\forall_n \forall_y \exists_x x^{2n - 1} = y,$$

(which we shall prove in the same way) shows that the operator '$\sqrt[2n-1]{\ }$' can be applied to <u>each</u> real number, nonnegative or not.]

EXPLORATION EXERCISES

<u>A</u>. Consider the function f defined by:

$$f(x) = \begin{cases} x, & x \leq 2 \\ \frac{1}{2}x + 1, & 2 < x \leq 6 \\ 2x - 8, & x > 6 \end{cases}$$

Complete.

1. f(−8) = ____ 2. f(75) = ____ 3. f(4.44) = ____

4. f(____) = 1.99 5. f(2.02) = ____ 6. f(____) = 2.01

7. f(3) = ____ 8. f(4) = ____ 9. f(5) = ____

10. f(201) = ____ 11. f(54) = ____ 12. f(9) = ____

13. f(7) = ____ 14. f(6.5) = ____ 15. f(6.2) = ____

16. f(20) = ____ 17. f(____) = 20 18. f(____) = 3

[As a partial check on your computations, compare your completed Exercises 18 and 8.]

19. $f(2) =$ _____ 20. $f(2.2) =$ _____ 21. $f(2.4) =$ _____

22. $f(2.6) =$ _____ 23. $f($_____$) = 2.4$ 24. $f($_____$) = 2.5$

[Compare your completed Exercises 24 and 7.]

25. $f($_____$) = 2.6$ 26. $f($_____$) = 2.8$ 27. $f($_____$) = 3.2$

28. $f($_____$) = 3.3$ 29. $f($_____$) = 3.4$ 30. $f($_____$) = 3.5$

31. $f($_____$) = 3.7$ 32. $f($_____$) = 3.9$ 33. $f($_____$) = 4.1$

[Compare your completed Exercises 33 and 15.]

34. The domain of f is __________________.

35. The range of f is __________________.

B. Consider the function g defined by:

$$g(x) = \begin{cases} x, & x \leq 2 \\ 3x - 4, & 2 < x \leq 6 \\ x + 10, & x > 6 \end{cases}$$

Complete.

1. $g(-8) =$ _____ 2. $g(4) =$ _____ 3. $g(5.9) =$ _____

4. $g(6.1) =$ _____ 5. $g(6.3) =$ _____ 6. $g(6.5) =$ _____

7. $g(1.99) =$ _____ 8. $g($_____$) = 1.99$ 9. $g(2) =$ _____

10. $g(2.01) =$ _____ 11. $g(3) =$ _____ 12. $g($_____$) = 5$

13. $g(4.2) =$ _____ 14. $g($_____$) = 6$ 15. $g(\frac{10}{3}) =$ _____

16. $g($_____$) = 8$ 17. $g($_____$) = 8.6$ 18. $g($_____$) = 9$

19. $g($_____$) = 11$ 20. $g($_____$) = 11.6$ 21. $g($_____$) = 11.9$

22. g(____) = 12.5   23. g(____) = 13.1   24. g(____) = 13.7

25. g(____) = 14.3   26. g(____) = 14.9   27. g(____) = 15.5

28. g(____) = 16.1   29. g(____) = 16.3   30. g(____) = 16.5

[Compare Exercises 4, 5, 6 with Exercises 25, 26, 27 and with Exercises 28, 29, 30.]

31. The domain of g is ________________.

32. The range of g is ________________.

C. If you have not yet done so, draw the graphs of the functions f and g of Parts A and B. [Use a separate sheet of paper for each graph.]

1. (a) Is each real number an argument of f? That is, is it the case that, for each real number x, there exists a real number y such that f(x) = y? Justify your answer in terms of the definition of f and also in graphical terms.

   (b) Is each real number a value of f? That is, is it the case that, for each real number y, there is a real number x such that f(x) = y? Justify your answer.

   (c) True or false?

   (1) $\forall_x \exists_y\ f(x) = y$ (2) $\forall_x \exists_y\ f(y) = x$ (3) $\forall_y \exists_x\ f(x) = y$

   (4) $\exists_x \forall_y\ f(x) = y$ (5) $\exists_y \forall_x\ f(x) = y$ (6) $\forall_x \forall_y\ f(x) = y$

2. (a) Now look at the graph, and the definition, of g. Is each real number an argument of g? That is, is it the case that

$$\forall_x \exists_y\ y = g(x)\ ?$$

   Justify your answer.

   (b) True or false?

$$\forall_y \exists_x\ y = g(x)$$

   (c) Which of the exercises in Part B justify your answer to (b)?

   (d) Interpret your answer to (b) in terms of the graph of g.

D. [Refer to the graph, and the definition, of f, concentrating your attention on the point (4, f(4)).]

1. Complete.

(a) $f(3.4) =$ ____ (b) $f(4) =$ ____ (c) $f(4.6) =$ ____

(d) $\{t: f(t) > 2.7\} = \{t: t >$ ____ $\}$

(e) $\{t: f(t) < 3.3\} = \{t: t <$ ____ $\}$

2. Verify the following statement:

$$\{t: 2.7 < f(t) < 3.3\} = \{t: 3.4 < t < 4.6\}$$

3. True or false?

(a) For each x, $2.7 < f(x) < 3.3$ <u>if and only if</u> $3.4 < x < 4.6$.

(b) For each x, if $3.4 < x < 4.6$ then $2.7 < f(x) < 3.3$.

(c) $\forall_x [4 - 0.6 < x < 4 + 0.6 \Rightarrow 3 - 0.3 < f(x) < 3 + 0.3]$

(d) $\forall_x [-0.6 < x - 4 < 0.6 \Rightarrow -0.3 < f(x) - f(4) < 0.3]$

(e) $\forall_x [\,|x - 4| < 0.6 \Rightarrow |f(x) - f(4)| < 0.3]$ [See Fig. 1.]

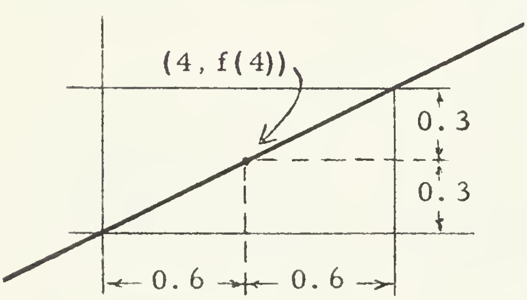


Fig. 1

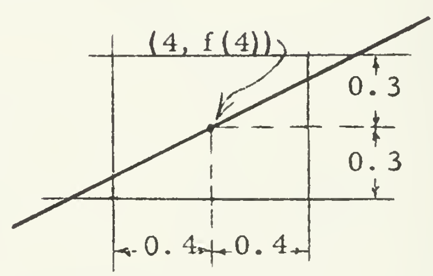


Fig. 2

(f) $\forall_x [\,|x - 4| < 0.4 \Rightarrow |x - 4| < 0.6]$

(g) $\forall_x [\,|x - 4| < 0.4 \Rightarrow |f(x) - f(4)| < 0.3]$ [See Fig. 2.]

(h) $\forall_x [\,|x - 4| < 0.01 \Rightarrow |f(x) - f(4)| < 0.3]$

(i) $\forall_x [\,|x - 4| < 0.6 \Rightarrow |f(x) - f(4)| < 157]$

4. Complete.

(a) $f(4) =$ ____ (b) $f($____$) = 3 - 0.07$ (c) $f($____$) = 3 + 0.07$

5. Verifying the following statement:

$$\{t: 2.93 < f(t) < 3.07\} = \{t: 3.86 < t < 4.14\}$$

6. True or false?

(a) For each x, if $3.86 < x < 4.14$ then $2.93 < f(x) < 3.07$.

(b) $\forall_x [-0.14 < x - 4 < 0.14 \Rightarrow -0.07 < f(x) - f(4) < 0.07]$

(c) $\forall_x [\,|x - 4| < 0.14 \Rightarrow |f(x) - f(4)| < 0.07]$

(d) $\forall_x [\,|x - 4| < 0.1 \Rightarrow |f(x) - f(4)| < 0.07]$

7. Complete.

$$\forall_x [\,|x - 4| < \qquad \Rightarrow |f(x) - f(4)| < 0.01]$$

[Hint. Compare this with Exercises 3(e) and 6(c).]

8. Verify your answer to Exercise 7.

9. Complete.

(a) $\forall_x [\,|x - 4| < \qquad \Rightarrow |f(x) - f(4)| < 0.002]$

(b) $\forall_x [\,|x - 4| < \qquad \Rightarrow |f(x) - f(4)| < 10^{-20}]$

(c) $\forall_x [\,|x - 4| < \qquad \Rightarrow |f(x) - f(4)| < 10^{-100}]$

(d) $\forall_x [\,|x - 4| < \qquad \Rightarrow |f(x) - f(4)| < 1.5]$

(e) $|6.5 - 4| =$

(f) $f(6.5) =$

(g) $|f(6.5) - f(4)|$? 1.5 [=, <, or > ?]

Compare!

True or false? [Draw figures like those in Exercise 3.]

(h) $\forall_x [\,|x - 4| < 3 \Rightarrow |f(x) - f(4)| < 1.5]$

(i) $\forall_x [\,|x - 4| < 2.9 \Rightarrow |f(x) - f(4)| < 1.5]$

(j) $\forall_x [\,|x - 4| < 2.7 \Rightarrow |f(x) - f(4)| < 1.5]$

(k) $\forall_x [\, |x - 4| < 2.5 \Rightarrow |f(x) - f(4)| < 1.5]$

(ℓ) $\forall_x [\, |x - 4| < 2.5 \Rightarrow |f(x) - f(4)| < 1.49]$

(m) $\forall_x [\, |x - 4| < 2 \Rightarrow |f(x) - f(4)| < 1.49]$

(n) $\forall_x [\, |x - 4| < 2 \Rightarrow |f(x) - f(4)| < 1.0001]$

10. (a) Choose a <u>very</u> small positive number c. Find a positive number d such that

$$\forall_x [\, |x - 4| < d \Rightarrow |f(x) - f(4)| < c].$$

(b) Choose a number $c_1 > c$. With the <u>same</u> positive number d you found in part (a), is it the case that

$$\forall_x [\, |x - 4| < d \Rightarrow |f(x) - f(4)| < c_1]?$$

(c) Write a formula for computing a suitable number d, given c, in case

(i) $0 < c \leq 1$; d =

(ii) $c > 1$; d = [See Exercise 9(n).]

11. True or false?

(a) For each positive number c [no matter how small], there is a positive number d such that

$$\forall_x [\, |x - 4| < d \Rightarrow |f(x) - f(4)| < c].$$

(b) $\forall_{c>0} \exists_{d>0} \forall_x [\, |x - 4| < d \Rightarrow |f(x) - f(4)| < c]$

<u>E</u>. [Refer, again, to the graph and definition of f, concentrating your attention, now, on the point (6, f(6)).]

1. Complete.

(a) f(6) = _____ (b) f(_____) = 4 - 0.2 (c) f(_____) = 4 + 0.2

2. Verify the following statement:

$$\{t: |f(t) - 4| < 0.2\} = \{t: 5.6 < t < 6.1\}$$

3. True or false?

(a) For each x, if $5.6 < x < 6.1$ then $|f(x) - f(6)| < 0.2$.

(b) $\forall_x [-0.4 < x - 6 < 0.1 \Rightarrow |f(x) - f(6)| < 0.2]$

(c) $\forall_x [-0.4 < x - 6 < 0.4 \Rightarrow -0.4 < x - 6 < 0.1]$

(d) $\forall_x [-0.1 < x - 6 < 0.1 \Rightarrow -0.4 < x - 6 < 0.1]$

(e) $\forall_x [\,|x - 6| < 0.1 \Rightarrow |f(x) - f(6)| < 0.2]$

4. Complete.

$$\forall_x [\,|x - 6| < \qquad\qquad\qquad \Rightarrow |f(x) - f(6)| < 2 \cdot 10^{-12}]$$

5. True or false?

$$\forall_{c>0} \exists_{d>0} \forall_x [\,|x - 6| < d \Rightarrow |f(x) - f(6)| < c]$$

6. Do you see that the statement in Exercise 5 says the same thing as:

$$\forall_{c>0} \exists_{d>0} \forall_h [\,|h| < d \Rightarrow |f(6 + h) - f(6)| < c]?$$

[Hint. If you replace the 'x - 6' in the first statement by 'h', what should you put in place of the 'x' in 'f(x)'?]

7. (a) True or false? [Look at the statement in Exercise 6.]
Starting at the argument 6 of the function f, any argument-change which is sufficiently small results in as small a value-change as you wish.

(b) Complete.
Suppose that someone is given a positive number c and, starting at the argument 6 of f, he wishes to change the value of f by at least c. Then, there is a positive number d such that he must change the argument by ____________.

F. [Refer, this time, to the graph, and the definition, of the function g of Part B, concentrating your attention on the point (4, g(4)).]

True or false?

1. $\forall_{c>0} \exists_{d>0} \forall_x [\,|x - 4| < d \Rightarrow |g(x) - g(4)| < c]$

2. Starting at 4, any sufficiently small change in the argument of g results in an arbitrarily small change in the value of g.

G. [This time, concentrate on the point $(6, g(6))$.]

1. Complete.

   (a) $g(6) = \_\_\_\_$ (b) $g(\_\_\_\_) = 14 - 3$ (c) $g(\_\_\_\_) = 14 + 3$

2. Verify.
$$\{t: |g(t) - 14| < 3\} = \{t: 5 < t < 7\}$$

3. True or false?

   (a) $\forall_x[\,|x - 6| < 1 \Rightarrow |g(x) - g(6)| < 3]$

   (b) $\exists_{d>0}\forall_x[\,|x - 6| < d \Rightarrow |g(x) - g(6)| < 3]$

4. Complete.

   (a) $g(6) = \_\_\_\_$ (b) $g(\_\_\_\_) = 14 - 0.3$ (c) $g(\_\_\_\_) = 14 + 0.3$

5. Verify.
$$\{t: |g(t) - g(6)| < 0.3\} = \{t: 5.9 < t \le 6\}$$

6. True or false?

   (a) $\forall_x[5.9 < x \le 6 \Rightarrow |g(x) - g(6)| < 0.3]$

   (b) $\forall_x[5.9 < x < 6.1 \Rightarrow |g(x) - g(6)| < 0.3]$

   (c) $\forall_x[\,|x - 6| < 0.1 \Rightarrow |g(x) - g(6)| < 0.3]$

   (d) $\forall_{x \le 6}[\,|x - 6| < 0.1 \Rightarrow |g(x) - g(6)| < 0.3]$

7. True or false?

   (a) $\exists_{d>0}\forall_x[\,|x - 6| < d \Rightarrow |g(x) - g(6)| < 0.3]$

   (b) $\exists_{d>0}\forall_x[\,|x - 6| < d \Rightarrow |g(x) - g(6)| < 10^{-300}]$

   (c) $\exists_{d>0}\forall_x[\,|x - 6| < d \Rightarrow |g(x) - g(6)| < 1.9]$

   (d) $\exists_{d>0}\forall_x[\,|x - 6| < d \Rightarrow |g(x) - g(6)| < 2]$

   (e) $\exists_{d>0}\forall_x[\,|x - 6| < d \Rightarrow |g(x) - g(6)| < 2.0001]$

   (f) Starting at 6, there are arbitrarily small changes in the argument of g which result in changing the value of g by at least 2.

H. Suppose that h is a function such that

$$h(x) = \begin{cases} \frac{1}{2}x, & \text{for } x \leq 2 \\ x + 1, & \text{for } 2 < x < 4 \\ -3x + 17, & \text{for } 4 \leq x \leq 5 \\ 1, & \text{for } 5 < x \leq 10. \end{cases}$$

1. What is the domain of h?

2. Draw a graph of h for the segment $\overline{-4, 10}$ of its domain.

3. True or false?

(a) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x + 1| < d \Rightarrow |h(x) - h(-1)| < c]$

(b) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 2| < d \Rightarrow |h(x) - h(2)| < c]$

(c) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 1.99| < d \Rightarrow |h(x) - h(1.99)| < c]$

(d) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 2.01| < d \Rightarrow |h(x) - h(2.01)| < c]$

(e) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 4| < d \Rightarrow |h(x) - h(4)| < c]$

(f) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 5| < d \Rightarrow |h(x) - h(5)| < c]$

(g) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 7| < d \Rightarrow |h(x) - h(7)| < c]$

(h) $\forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - 10| < d \Rightarrow |h(x) - h(10)| < c]$

4. True or false?

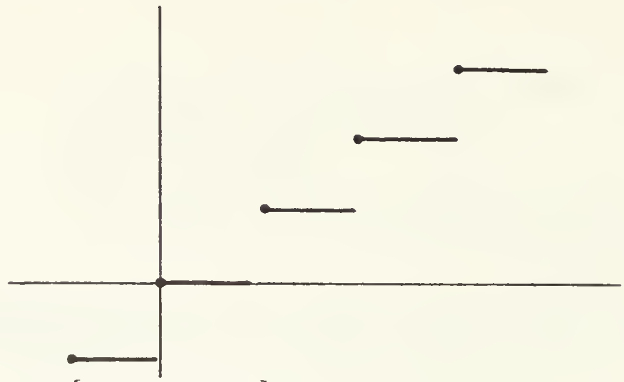
(a) $\forall_{x_o \leq 10} \forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - x_o| < d \Rightarrow |h(x) - h(x_o)| < c]$

(b) $\forall_{x_o < 2} \quad \forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - x_o| < d \Rightarrow |h(x) - h(x_o)| < c]$

(c) $\forall_{x_o \leq 10} \Big[(x_o \neq 2 \text{ and } x_o \neq 5) \Rightarrow$
$\qquad \forall_{c>0} \exists_{d>0} \forall_{x \leq 10} [\, |x - x_o| < d \Rightarrow |h(x) - h(x_o)| < c]\Big]$

(d) Starting at any argument of h other than 2 and 5, the values of h for arguments sufficiently close to the given argument will be arbitrarily close to the value of h at the given argument.

I. Suppose that f is the greatest integer function. Here is a graph of f.



1. Complete [if possible].

   (a) $\quad > 0$ and $\forall_x \left[\left| x - \frac{3}{2} \right| < \quad \Rightarrow \left| f(x) - f\left(\frac{3}{2}\right) \right| < \frac{1}{2} \right]$

   (b) $\quad > 0$ and $\forall_x \left[\left| x - 3 \right| < \quad \Rightarrow \left| f(x) - f(3) \right| < 1 \right]$

2. True or false?

   (a) $\forall_{c>0} \exists_{d>0} \forall_x \left[\left| x - \frac{3}{2} \right| < d \Rightarrow \left| f(x) - f\left(\frac{3}{2}\right) \right| < c \right]$

   (b) $\forall_{c>0} \exists_{d>0} \forall_x \left[\left| x - 3 \right| < d \Rightarrow \left| f(x) - f(3) \right| < c \right]$

   (c) $\forall_{x_o} \forall_{c>0} \exists_{d>0} \forall_x \left[\left| x - x_o \right| < d \Rightarrow \left| f(x) - f(x_o) \right| < c \right]$

3. Complete.

   $\forall_{x_o \notin \quad} \forall_{c>0} \exists_{d>0} \forall_x \left[\left| x - x_o \right| < d \Rightarrow \left| f(x) - f(x_o) \right| < c \right]$

   ↑
   ?

CONTINUOUS FUNCTIONS

In the preceeding exercises you have seen that, given a function f and an argument a_o of f, it may be the case that, starting at a_o,

> any sufficiently small argument-change
> results in an arbitrarily small value-change

--that is,

$$\forall_{c>0} \exists_{d>0} \forall_{x \in \mathfrak{D}_f} \left[\left| x - a_o \right| < d \Rightarrow \left| f(x) - f(a_o) \right| < c \right].$$

If [and only if] that is the case, the function f is said to be continuous at a_o.

<u>Definition</u>.

A function f is continuous at a_o if and only if $a_o \in \mathcal{D}_f$ and $\forall_{c>0} \exists_{d>0} \forall_{x \in \mathcal{D}_f} [\, |x - a_o| < d \Rightarrow |f(x) - f(a_o)| < c]$.

A function is continuous [merely] if and only if it is continuous at each of its arguments.

For example, look back at Exercise 2 of Part I. Since each number is an argument of the greatest integer function, an answer 'true' for (a) is equivalent to asserting that the greatest integer function is continuous at 3/2; and an answer 'false' for (b) is equivalent to asserting that the greatest integer function is not continuous [or: is <u>discontinuous</u>] at 3.

For another example, look back at the graph you drew in answer to Exercise 2 of Part H. Is the function h continuous at −1? At 2? At 2.01? At 4? At 5? At 7? At 10? Compare your answers with those you gave for parts (a), (b), (d), (e), (f), (g), and (h) of Exercise 3. Is the function h continuous at 11? Describe the set of those real numbers at which h is continuous.

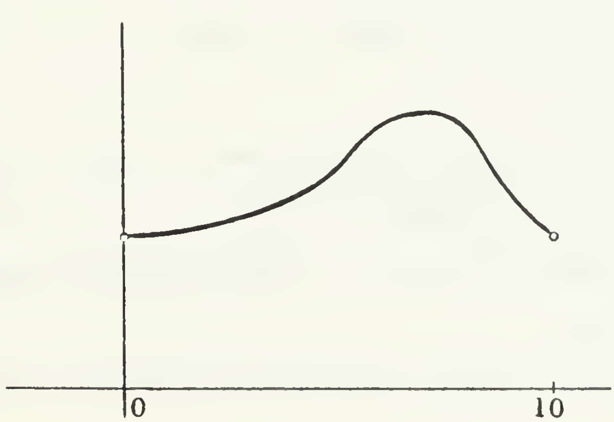
Is each constant function whose domain is the set of all real numbers continuous?

Is each linear function continuous?

Is each function which is a subset of a continuous function continuous?

EXERCISES

<u>A</u>. Here is a graph of a function f whose domain is $\{x: 0 < x < 10\}$.



1. As you should be able to see from the graph, f is continuous at 2. Say this in two other ways. [Hint. See Exercises 1 and 2 of Part F on page 9-207.]

   (a) $\forall_{c>0}\exists_{d>0}\forall_{x \in \mathcal{D}_f}[$

   (b) Starting at . . .

2. State in two ways that f is continuous at 6.

3. State in two ways that f is continuous.

4. Tell why the following statements are false.

   (a) $\forall_{c>0}\forall_{d>0}\forall_{x \in \mathcal{D}_f}[\,|x - 2| < d \Rightarrow |f(x) - f(2)| < c]$

   (b) $\exists_{d>0}\forall_{c>0}\forall_{x \in \mathcal{D}_f}[\,|x - 2| < d \Rightarrow |f(x) - f(2)| < c]$

5. (a) State in two ways that sq is continuous.

   (b) State in two ways that sq^+ is continuous.

   (c) Explain why, if sq is continuous at 2, sq^+ must be.

   (d) Explain why, if sq is continuous, sq^+ must be.

B. 1. Suppose that f is such that $f(3) = 5$ and, for $x \neq 3$, $f(x) = 2x$.

   (a) Graph the function f.

   (b) True or false?

      (i) For each x_o, f is continuous at x_o.

      (ii) f is continuous at 3.

   (c) In part (b), you should have answered 'false' for (ii) and, consequently [Explain.], for (i).

      (i) Put a restriction on 'x_o' in (i) of part (b) to obtain a true statement.

      (ii) Change the definition of f at 3 so that (ii) of part (b) will be true.

2. The signum function is defined by:

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1, & \text{for } x > 0 \end{cases}$$

(a) Graph sgn.

(b) True or false?

(i) sgn is continuous.

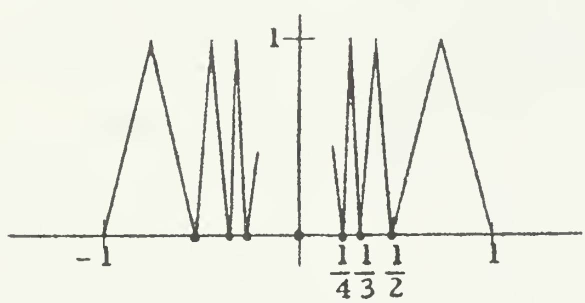
(ii) sgn is continuous at 0.

(c) In part (b), you should have answered 'false' for (ii) and, consequently [Explain.], 'false' for (i).

(i) Complete: sgn is continuous except at ____.

(ii) Can you change the definition of sgn at 0 so that (ii) of part (b) will be true?

3. Here is part of the graph of a function g whose domain is $\{x: |x| \leq 1\}$.



The whole graph consists of the origin and the points on the legs of an infinite number of isosceles triangles. [The base of each triangle is one of the segments $-\frac{1}{n}, -\frac{1}{n+1}$ or one of the segments $\frac{1}{n+1}, \frac{1}{n}$, for any n, and the altitude of each triangle is 1.] By definition, $g(0) = 0$.

True or false?

(a) $\forall_{c>0} \exists_{d>0} \forall_{x \in \mathfrak{D}_f} [\, |x| < d \Rightarrow |g(x)| < c]$

(b) g is not continuous at 0.

(c) g is continuous at each point of its domain except 0.

* * *

Your knowledge of absolute values [Theorem 169 or Parts N and O on pages 9-15 through 9-18] can be used in proving functions to be continuous.

Example. Show that the function f is continuous, where

$$f(x) = -3x + 2, \text{ for all } x.$$

Solution. Given any number a_o and any positive number c, the job is to find a positive number d such that

$$\forall_x [\,|x - a_o| < d \Rightarrow |f(x) - f(a_o)| < c].$$

Now, for any a,

$$\begin{aligned} |f(a) - f(a_o)| &= |(-3a + 2) - (-3a_o + 2)| \\ &= |-3(a - a_o)| \\ &= |-3| \cdot |(a - a_o)|. \end{aligned} \quad \text{[Theorem 169a]}$$

So, $|f(a) - f(a_o)| < c$ if $|-3| \cdot |a - a_o| < c$.

Since $|-3| \cdot |a - a_o| < c$ if $|a - a_o| < c/|-3|$, it follows that

$$|f(a) - f(a_o)| < c \quad \text{if } |a - a_o| < c/|-3|.$$

Since $|-3| > 0$, it follows that, for $c > 0$, $c/|-3| > 0$.

Consequently, for each $c > 0$, there is a number $d > 0$ [$c/|-3|$ is one such number] such that

$$\forall_x [\,|x - a_o| < d \Rightarrow |f(x) - f(a_o)| < c].$$

Hence, f is continuous at any number a_o--that is, f is continuous.

* * *

C. 1. Suppose that, for some numbers $m \neq 0$ and b,

$$f(x) = mx + b, \text{ for all } x.$$

Prove that f is continuous. [That is, prove that each __?__ function is continuous.] [Hint. Follow the Example above.]

2. Suppose that g is the absolute value function. [Graph it.]

Prove:

$$\forall_{x_o} \forall_{c>0} \exists_{d>0} \forall_x [\,|x - x_o| < d \Rightarrow |g(x) - g(x_o)| < c]$$

--that is, prove that g is continuous.

[Hint. Note that

$$\forall_x \forall_y \,\big|\,|x| - |y|\,\big| \leq |x - y|.\ ]$$

3. Consider the principal square-rooting function h:

$$h(x) = \sqrt{x}, \text{ for } x \geq 0$$

(a) Graph h.

(b) Complete [by putting the same expression in both blanks].

$$\forall_{c>0} \Big(\qquad > 0 \text{ and } \forall_{x\geq 0} [\,|x - 7| < \qquad \Rightarrow |h(x) - h(7)| < c] \Big)$$

[Hint. Notice that, for any $a \geq 0$,

$$|a - 7| = |(\sqrt{a} + \sqrt{7})(\sqrt{a} - \sqrt{7})| = |\sqrt{a} + \sqrt{7}| \cdot |\sqrt{a} - \sqrt{7}|$$

$$= (\sqrt{a} + \sqrt{7})\,|\sqrt{a} - \sqrt{7}| \geq \sqrt{7}\,|\sqrt{a} - \sqrt{7}| \quad \text{[Explain.]}$$

and, so,

$$|h(a) - h(7)| \leq \frac{|a - 7|}{\sqrt{7}}\ .]$$

(c) True or false?

(i) The function h is continuous at 7.

(ii) The function h is continuous [that is, h is continuous at each $x_o \geq 0$].

(d) Support your answer for (ii) of part (c). [Hint. You may want to consider the cases $x_o > 0$ and $x_o = 0$ separately.]

4. Consider the squaring function sq:

$$sq(x) = x^2, \text{ for all } x$$

(a) Graph sq.

(b) Do you think that sq is continuous?

*

Let's try to prove that sq is continuous at, say, 2--that is, that, given any $c > 0$, we can find a $d > 0$ such that

$$\forall_x[\,|x - 2| < d \Rightarrow |sq(x) - sq(2)| < c].$$

Now, for any a,

$$sq(a) - sq(2) = a^2 - 2^2 = (a + 2)(a - 2)$$

and, so,

$$|sq(a) - sq(2)| = |a + 2| \cdot |a - 2| \quad \text{[Explain.]}$$

and

$$|sq(a) - sq(2)| < c \text{ if } |a + 2| \cdot |a - 2| < c$$

--that is,

$$\text{if } |a - 2| < \frac{c}{|a + 2|}.$$

One's first impulse, here, is likely to be to say that the problem is finished--for any $c > 0$, we can take for d the number $c/|a + 2|$. The trouble with this is that, given a number c, the value of the expression '$c/|a + 2|$' depends on the value we choose for 'a'. And, what we need is one number $d > 0$ such that, for <u>any</u> a,

$$|sq(a) - sq(2)| < c \ \underline{\text{if}}\ \ |a - 2| < d.$$

If we could find a positive number d such that, for any a,

$$d \le \frac{c}{|a + 2|},$$

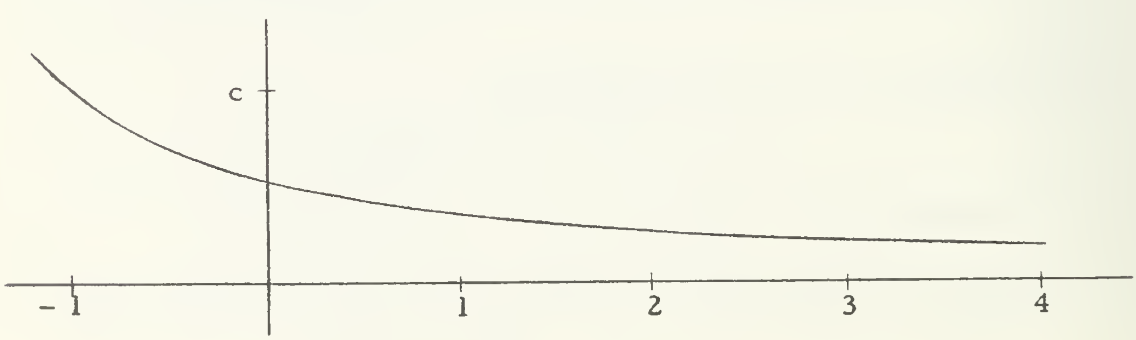
we'd be all set. For, for such a $d > 0$,

$$\text{if } |a - 2| < d \text{ then } |a - 2| < \frac{c}{|a + 2|}$$

and, as we have seen,

$$\text{if } |a - 2| < \frac{c}{|a + 2|} \text{ then } |sq(a) - sq(2)| < c.$$

To see what is happening, let's graph the values of '$c/|a + 2|$'.



[Here is a table used in drawing the graph:

a	-1	0	1	2	3	4
$c/\lvert a+2\rvert$	c	$c/2$	$c/3$	$c/4$	$c/5$	$c/6$

Note that, since we have not indicated the scale on the vertical axis, the same graph will do for any number $c > 0$.]

Evidently, for larger values of 'a', one gets smaller values of '$c / |a + 2|$'--in fact, if a is sufficiently large, $c / |a + 2|$ is an arbitrarily small positive number. It turns out that there is no $d > 0$ such that, for any a,

$$d \leq \frac{c}{|a+2|} .$$

However, there is a way out. We aren't interested in all numbers a--just in those near 2. And, from the graph, if, say, $|a - 2| < 1$ [that is, if $1 < a < 3$],

$$\frac{c}{5} \leq \frac{c}{|a+2|} .$$

So, if $|a - 2| < 1$ and $|a - 2| < c/5$ then $|sq(a) - sq(2)| < c$. Hence, for d, we can take the smaller of the two positive numbers 1 and $c/5$. If the number c we choose is greater than 5, we take d to be 1; if not, we take d to be $c/5$. So, in any case, if $c > 0$ then there is a $d > 0$ such that

$$\forall_x [\, |x - 2| < d \Rightarrow |sq(x) - sq(2)| < c]$$

--that is, sq <u>is</u> continuous at 2.

Now, let's see how our knowledge of absolute values could have been used to show that this way of choosing d would work. Recall that we showed, rather easily, that

$$|sq(a) - sq(2)| < c \text{ if } |a + 2| \cdot |a - 2| < c.$$

Now, $\quad |a + 2| \leq |a| + |2| \quad$ [What theorem?]

and, since $\quad |a| - |2| \leq |a - 2|, \quad$ [What theorem?]

$$|a| \leq |a - 2| + |2|,$$

and, so $\quad |a| + |2| \leq |a - 2| + 2|2|.$

Hence, $\quad |a + 2| \leq |a - 2| + 2|2|.$

Consequently, for $|a - 2| < 1$,

$$|a + 2| < 1 + 2|2|.$$

So, for $|a - 2| < 1$,

$$|a + 2| \cdot |a - 2| < (1 + 2|2|)|a - 2|.$$

Hence, for $|a - 2| < 1$,

if $(1 + 2|2|)|a - 2| < c$ then $|sq(a) - sq(2)| < c$

--that is,

if $|a - 2| < 1$ and $|a - 2| < \frac{c}{1 + 2|2|}$ then $|sq(a) - sq(2)| < c$.

So, again, we take for d the smaller of the numbers 1 and $\frac{c}{1+2|2|}$.

This procedure will work to show, for any number a_0, sq is continuous at a_0. [Just replace most of the '2's by 'a_0's.]

*

4. [continued]

☆(c) Prove that sq is continuous.

☆(d) Prove:

> Theorem 186
>
> Each positive-integral power function is continuous.

[Hint. One way to proceed is by induction. Clearly, the first positive-integral power function is continuous [Explain.].

Suppose, then, that the pth power function is continuous--say, at a_0. That is, assume that

$$(*)\ \forall_{c>0}\exists_{d>0}\forall_x[\,|x - a_0| < d \Rightarrow |x^p - a_0{}^p| < c].$$

Now, for any a,

$$a^{p+1} - a_0{}^{p+1} = (a^p - a_0{}^p)a + (a - a_0)a_0{}^p.$$

So,

$$\begin{aligned} |a^{p+1} - a_0{}^{p+1}| &= |(a^p - a_0{}^p)a + (a - a_0)a_0{}^p| \\ &\le |(a^p - a_0{}^p)a| + |(a - a_0)a_0{}^p| \\ &= |a^p - a_0{}^p| \cdot |a| + |a - a_0| \cdot |a_0{}^p|. \end{aligned} \quad \begin{matrix} \text{[Explain.]} \\ \text{[Explain.]} \end{matrix}$$

As in the case of sq, for $|a - a_o| < 1$, $|a| < 1 + |a_o|$. Hence, for $|a - a_o| < 1$,

$$|a^{p+1} - a_o^{p+1}| < |a^p - a_o^p|(1 + |a_o|) + |a - a_o| \cdot |a_o^p|.$$

So, it will follow that

$$|a^{p+1} - a_o^{p+1}| < c$$

if

$$\left.\begin{array}{l} (1)\ |a - a_o| < 1, \\ (2)\ |a^p - a_o^p|(1 + |a_o|) < \frac{c}{2}, \\ \text{and } (3)\ |a - a_o| \cdot |a_o^p| < \frac{c}{2}. \end{array}\right\} \text{[Explain.]}$$

Now, looking at (2),

$$\frac{c}{2(1 + |a_o|)} \text{ is a positive number}$$

and, so, by the inductive hypothesis (*), there is a positive number --say, d_1--such that

$$\text{if } |a - a_o| < d_1 \text{ then } |a^p - a_o^p| < \frac{c}{2(1 + |a_o|)}.$$

So, if $|a - a_o| < d_1$ then $|a^p - a_o^p|(1 + |a_o|) < \frac{c}{2}$.

As to (3), we see that

$$\text{if } |a - a_o| < \frac{c}{2|a_o^p|} \text{ then } |a - a_o| \cdot |a_o^p| < \frac{c}{2}.$$

Consequently, if d is the smallest of the positive numbers 1, d_1, and $\frac{c}{2|a_o^p|}$, it follows that

$$\forall_x [\, |x - a_o| < d \Rightarrow |x^{p+1} - a_o^{p+1}| < c].$$

So, assuming (*)--that is, assuming that the pth power function is continuous at a_o--it follows that the (p + 1)th power function is continuous at a_o. So, if the pth power function is continuous then the (p + 1)th power function is continuous.

This completes part (ii) of the inductive proof. From this and part (i), it follows that each positive-integral power function is continuous.]

A THEOREM ON CONTINUOUS MONOTONIC FUNCTIONS

One of our problems, as formulated on page 9-201 [just before the Exploration Exercises which led to the notion of continuity] was to find a way of showing that for each n, the range of the nth power function includes all nonnegative numbers--more precisely, that

$$\forall_n \forall_{y \geq 0} \exists_{x \geq 0}\, x^n = y.$$

As pointed out there, we need this result in order to know that, for each n, the domain of the principal nth root function is the set of all nonnegative numbers.

More generally, our problem is to discover a method for showing, of as many numbers as we can, that they belong to the range of a given function f. That is, given a function f, we wish to have a method for proving existence theorems like:

$$\forall_{y \in R} \exists_{x \in \mathcal{D}_f} f(x) = y \qquad \text{[See (I) on page 9-193.]}$$

for as "large" a set R as possible. Actually, our present interest is in functions which have inverses--that is, such that

$$\forall_{x_1 \in \mathcal{D}_f} \forall_{x_2 \in \mathcal{D}_f} [f(x_1) = f(x_2) \Rightarrow x_1 = x_2] \qquad \text{[See (II) on page 9-193.]}$$

and, more particularly, in monotonic functions.

So, we might reformulate our problem as follows:

(P) Find a method for determining the range of a monotonic function.

If we solved this problem, we should have a method for determining the domains of the inverses of monotonic functions. Since there are many monotonic functions which--like the positive-integral power functions, restricted to nonnegative arguments--have useful inverses, this would be a good thing to be able to do.

As a matter of fact, all the monotonic functions which will interest us will--again like the [restricted] power functions--be continuous. Moreover, the domain of each of these functions will be either the set of all real numbers, or a ray [for example, $\{x: x \geq 0\}$], or a half-line [$\{x: x > 0\}$], or an interval [$\{x: |x| < 1\}$], or a segment [$\{x: |x| \leq 1\}$]. So, it will be sufficient to solve a subcase of the problem (P):

(P_c) Find a method for determining the range of a continuous monotonic function whose domain is either the set of all real numbers, or a ray, or a half-line, or an interval, or a segment.

Since what can be done for increasing functions can be done for decreasing functions, we shall begin by considering continuous increasing functions. And, because it is the simplest case, we shall consider, at first, continuous increasing functions whose domains are segments. It will turn out that once we have solved this case of problem (P_c), each of the remaining cases will be easy to handle. As a dividend, we shall discover that the inverse of each function of the kinds described in (P_c) is not only--as we already know--monotonic, but is, itself, continuous.

To begin with, consider the functions g and h whose graphs are shown in Fig. 1 and Fig. 2. Both are increasing functions, each has the segment $\overline{a, b}$ as its domain, and $g(a) = c = h(a)$ and $g(b) = d = h(b)$.

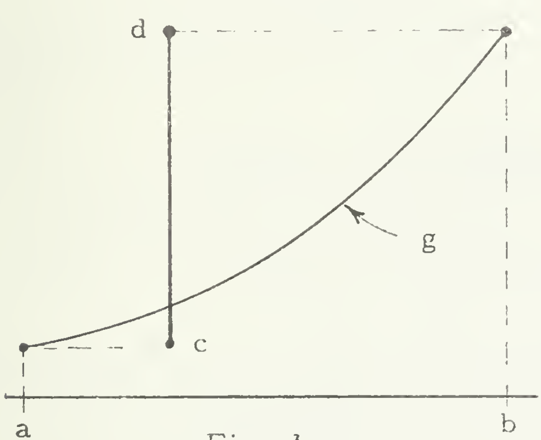


Fig. 1

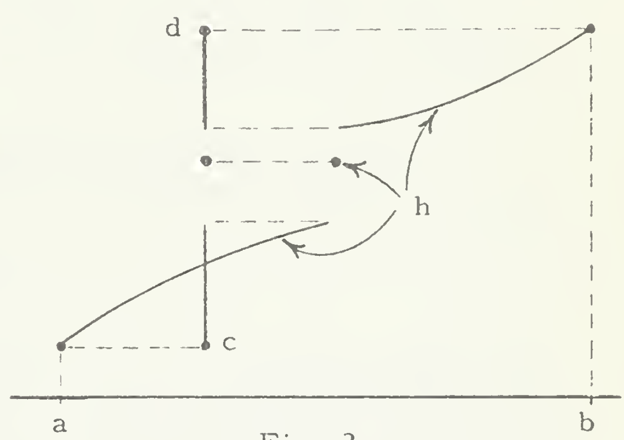


Fig. 2

Since both functions are monotonic, it follows that the range of each is a subset of the segment $\overline{c, d}$. But, the range of g is the whole of this segment, while the range of h is only part of this segment. What difference between g and h do you think accounts for this difference in their ranges?

It is easy to see that if f is any increasing function whose domain is the segment $\overline{a, b}$ then the range of f is a subset of the segment $\overline{f(a), f(b)}$. And, as you have seen, the range may or may not be the whole of this

segment. For simplicity, if the range is the whole segment, let us say that the function f has maximal range. [Why 'maximal'?] The examples g and h may have suggested to you that

> an increasing function whose domain is a segment
> has maximal range if and only if it is continuous.

This is the case. We shall prove it to be so, beginning with the if-part. This part amounts to the following statement:

(a) A continuous increasing function whose domain is a segment has maximal range.

Before proving (a), let's illustrate how it will help us in solving our original problem. This was, you remember, to show that each non-negative number has, for each n, a nonnegative nth root. We shall illustrate, in a small way, the use of (a) by using it to prove that the nonnegative number 5 has a nonnegative 4th root--that is, that there is a nonnegative number whose 4th power is 5. This is, now, very easy to do:

> From Theorem 186 we know that the 4th power function is continuous. Consequently, the restricted power function f, where
>
> $$f = \{(x,\ y),\ x \geq 0 : y = x^4\},$$
>
> is also a continuous function. Furthermore, by Theorem 185, this function is increasing. Consider, now, the subset f_o, where
>
> $$f_o = \{(x,\ y),\ 0 \leq x \leq 2 : y = x^4\}.$$
>
> As a subset of the continuous increasing function f, f_o is, also, a continuous increasing function. Moreover, the domain of f_o is a segment--the segment $\overleftrightarrow{0,\ 2}$. Consequently, by (a) [and the meaning of 'maximal range'], the range of f_o is the segment $\overleftrightarrow{f_o(0),\ f_o(2)}$. Since $f_o(0) = 0$ and $f_o(2) = 16$, it follows that the range of f_o is $\{y : 0 \leq y \leq 16\}$. Consequently, any number between 0 and 16 is the 4th power of some number between 0 and 2. Since 5 is between 0 and 16, 5 has a nonnegative 4th root.

It should now be clear how (a) [together with Theorems 185 and 186] can be used in proving that, for each n, each nonnegative number has a nonnegative nth root.

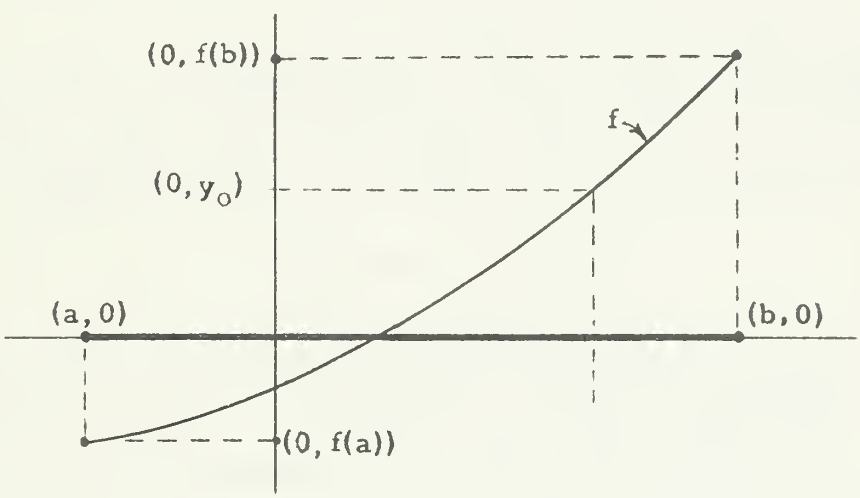
In proving that 5 has a nonnegative 4th root, we needed, besides (a) and Theorems 185 and 186, to know that there is a number whose 4th power is not less than 5. One such number is 2, and we used this in describing the function f_o. We could as well have used 1.5 [Explain.]. In proving the general result we shall need to know that

$$\forall_n \forall_{y \geq 0} \exists_{x \geq 0} x^n \geq y.$$

This follows easily from Bernoulli's Inequality [Theorem 162].

Now that we have seen how (a) can be used to solve (P_c), let's find a way to prove (a). [We shall see that the least upper bound principle will be of help to us.]

Consider a continuous increasing function f whose domain, $\mathcal{D}_f$, is a segment $\{x: a \leq x \leq b\}$. We know that the range of f is a subset of



the segment $\{y: f(a) \leq y \leq f(b)\}$ and what we wish to show is that all members of this segment are values of f. Since the end points of the segment are, by definition, values of f, what we need to show is that if $f(a) < y_o < f(b)$ then there is an x_o such that $a \leq x_o \leq b$ and $f(x_o) = y_o$. [Since f is increasing and $f(a) < y_o$, it will be the case that $a < x_o$.]

In order to see how to find such a number x_o--and, so, to prove (a)--let's suppose, for a moment, that there is such a number. Since f is an increasing function, it follows, since $f(x_o) = y_o$, that

$$\{x \in \mathcal{D}_f: x < x_o\} = \{x \in \mathcal{D}_f: f(x) < y_o\}.$$

Since, as noted previously, if there is such a number x_o, $a < x_o$, and since $a \in \mathcal{D}_f$, it follows that $\{x \in \mathcal{D}_f: x < x_o\} \neq \emptyset$. Hence, x_o is the least upper bound of $\{x \in \mathcal{D}_f: x < x_o\}$. Consequently, _if_ there is

an $x_0 \in \mathcal{D}_f$ such that $f(x_0) = y_0$ then x_0 must be the least upper bound of $\{x \in \mathcal{D}_f : f(x) < y_0\}$. It is this insight which we need in order to prove (a). We see that what we need to do, for $f(a) < y_0 < f(b)$, is to prove that

(i) $\{x \in \mathcal{D}_f : f(x) < y_0\}$ has a least upper bound which belongs to $\mathcal{D}_f$

and that, having done so, we should be able to prove that

(ii) the value of f at the least upper bound of $\{x \in \mathcal{D}_f : f(x) < y_0\}$ is y_0.

Of these two statements, (i) is an almost immediate consequence of the least upper bound principle. The proof of (ii) is very like the proof, in section 9.03, that the square of the least upper bound of $\{x \geq 0 : x^2 < 2\}$ is 2. In the latter proof we used the monotonicity of sq^+ and two generalizations [(Φ_1) and (Φ_2) on page 9-31] concerning a special function h. In the proof of (ii) we shall use the monotonicity of our function f [instead of the monotonicity of sq^+] and the continuity of f [instead of (Φ_1) and (Φ_2)].

Now, finally, here is a proof of (a):

Suppose that f is an increasing function whose domain is $\{x : a \leq x \leq b\}$ and that $f(a) < y_0 < f(b)$. Let $E = \{x \in \mathcal{D}_f : f(x) < y_0\}$. Since $a \in \mathcal{D}_f$, it follows that $a \in E$. So, E is nonempty. Since, by definition,

$$E \subseteq \mathcal{D}_f = \{x : a \leq x \leq b\},$$

it follows that b is an upper bound of E. Since E is nonempty and has an upper bound, it follows, from the ℓubp, that E has a least upper bound. Let x_0 be the least upper bound of E. Since $a \in E$ and x_0 is an upper bound of E, $a \leq x_0$. Since b is an upper bound of E and x_0 is the least upper bound of E, $x_0 \leq b$. Since $\mathcal{D}_f = \{x : a \leq x \leq b\}$, $x_0 \in \mathcal{D}_f$. [This completes the proof of (i).]

To prove that $f(x_0) = y_0$ it is sufficient [by Theorem 86a] to show that $f(x_0) \not< y_0$ and that $f(x_0) \not> y_0$.

Suppose that $f(x_0) < y_0$. It follows that if $c = y_0 - f(x_0)$ then $c > 0$. So, since f is continuous at x_0, there is a $d > 0$ such that

$$\forall_{x \in \mathcal{D}_f} [\,|x - x_0| < d \Rightarrow |f(x) - f(x_0)| < c].$$

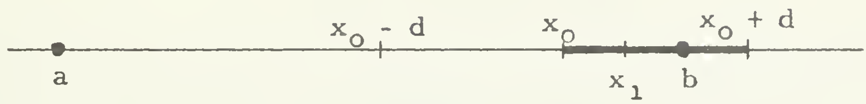
Now, for each x, $\quad x_0 < x < x_0 + d \Rightarrow |x - x_0| < d$

and

$$|f(x) - f(x_0)| < c \Rightarrow f(x) - f(x_0) < c.$$

Consequently, for each $x \in \mathfrak{D}_f$, it follows, since $c = y_0 - f(x_0)$, that

$$\text{if } x_0 < x < x_0 + d \text{ then } f(x) - f(x_0) < y_0 - f(x_0).$$



Now, since, by hypothesis, $y_0 < f(b)$ and, by assumption, $f(x_0) < y_0$, it follows that $f(x_0) < f(b)$ and, since f is increasing, that $x_0 < b$. Hence, there is an $x_1 \in \mathfrak{D}_f$ such that $x_0 < x_1 < x_0 + d$. [Both b and $x_0 + d$ are greater than x_0. For x_1 one might take the midpoint of the segment whose end points are x_0 and the smaller of the numbers b and $x_0 + d$.] As we have seen, for such a number x_1, $f(x_1) - f(x_0) < y_0 - f(x_0)$--that is, $f(x_1) < y_0$. Hence, by definition, $x_1 \in E$. So, x_1 is a member of E which is greater than x_0. But, this is impossible because x_0 is an upper bound of E. Hence, $f(x_0) \not< y_0$.

Suppose, now, that $f(x_0) > y_0$. It follows that if $c = f(x_0) - y_0$ then $c > 0$. So, since f is continuous at x_0, there is a $d > 0$ such that

$$\forall_{x \in \mathfrak{D}_f} [\, |x - x_0| < d \Rightarrow |f(x) - f(x_0)| < c].$$

Now, for each x,

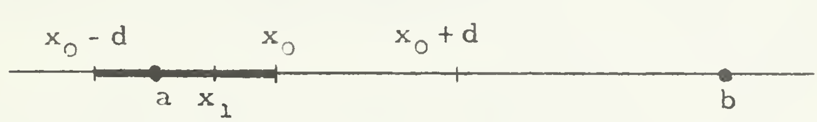
$$x_0 - d < x < x_0 \Rightarrow |x - x_0| < d$$

and

$$|f(x) - f(x_0)| < c \Rightarrow -c < f(x) - f(x_0).$$

Consequently, for each $x \in \mathfrak{D}_f$, it follows, since $-c = y_0 - f(x_0)$, that

$$\text{if } x_0 - d < x < x_0 \text{ then } y_0 - f(x_0) < f(x) - f(x_0).$$



Now, since, by hypothesis, $y_0 > f(a)$ and, by assumption, $f(x_0) > y_0$, it follows that $f(x_0) > f(a)$ and, since f is increasing, that $x_0 > a$. Hence, there is an $x_1 \in \mathfrak{D}_f$ such that $x_0 - d < x_1 < x_0$ [Explain.]. As we have seen, for such a number x_1, $y_0 - f(x_0) < f(x_1) - f(x_0)$--that is, $y_0 < f(x_1)$. Since, for each $x \in E$, $f(x) < y_0$, it follows that, for each $x \in E$, $f(x) < f(x_1)$ and, since f is increasing, $x < x_1$. So, x_1 is an upper bound of E which is less than x_0. But, this is impossible because x_0 is the least upper bound of E. Hence, $f(x_0) \not> y_0$.

So, we have proved (a).

Before applying (a) [in the way we have illustrated] to prove the existence theorem on page 9-220:

$$\forall_n \forall_{y>0} \exists_{x>0}\, x^n = y$$

let's recall that (a) was only the if-part of a conjecture you made on page 9-223. This conjecture was that an increasing function whose domain is a segment has maximal range if and only if it is continuous. To complete the proof of this conjecture we still have to prove:

(b) An increasing function whose domain is a segment and whose range is maximal is continuous.

[Aside from checking on our conjecture, this theorem will be useful to us because it [together with the if-part] will allow us to conclude at once that the inverse of a continuous increasing function whose domain is a segment is also continuous.]

Here is a proof of (b):

Suppose that f is an increasing function whose domain is the segment $\{x: a \le x \le b\}$ and whose range is maximal--that is, the range of f is $\{y: f(a) \le y \le f(b)\}$. We wish to show, for each $x_0 \in \mathcal{D}_f$, that, given $c > 0$, there is a $d > 0$ such that

$$\forall_{x \in \mathcal{D}_f} [\,|x - x_0| < d \Rightarrow |f(x) - f(x_0)| < c].$$

We can simplify the proof by recalling that a number d which "works" for a given number c also works for any larger number. So, it is enough to show how to find, for any sufficiently small positive number c, a number d which works for this c.

To begin with, suppose that $a < x_0 < b$. Then, since f is increasing, $f(a) < f(x_0) < f(b)$ and, hence, $f(x_0) - f(a)$ and $f(b) - f(x_0)$ are both positive. Let c_0 be the smaller of these positive numbers--so that $c_0 > 0$ and

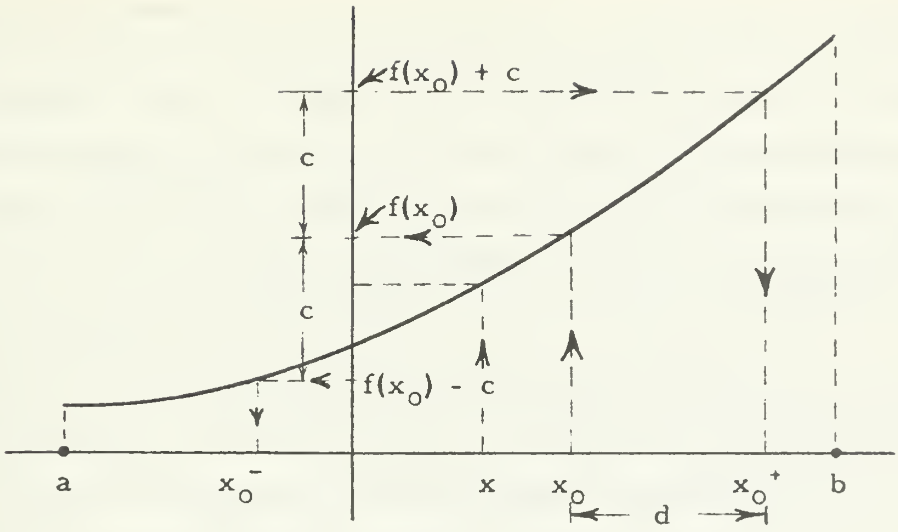
$$c_0 \le f(x_0) - f(a) \text{ and } c_0 \le f(b) - f(x_0).$$

We shall show how to find, for any c such that $0 < c \le c_0$, a number $d > 0$ which works for this c. For such a number c,

$$c \le f(x_0) - f(a) \text{ and } c \le f(b) - f(x_0)$$

--that is,

$$f(a) \le f(x_0) - c < f(x_0) < f(x_0) + c \le f(b).$$



Since the range of f is maximal, it follows that both $f(x_0) - c$ and $f(x_0) + c$ belong to the range of f--that is, there are arguments--say, x_0^- and x_0^+ --of f such that $f(x_0^-) = f(x_0) - c$ and $f(x_0^+) = f(x_0) + c$. Since f is increasing and since

$$f(x_0^-) < f(x_0) < f(x_0^+),$$

it follows that $x_0^- < x_0 < x_0^+$. Moreover, since f is increasing, it follows that, for each $x \in \mathfrak{D}_f$,

$$\text{if } x_0^- < x < x_0^+ \text{ then } f(x_0^-) < f(x) < f(x_0^+)$$

--that is,

$$\text{if } x_0^- < x < x_0^+ \text{ then } f(x_0) - c < f(x) < f(x_0) + c.$$

Since $x_0^- < x_0 < x_0^+$, it follows that both $x_0 - x_0^-$ and $x_0^+ - x_0$ are positive. Let d be the smaller of these numbers. Then $d > 0$ and

$$d \leq x_0 - x_0^- \quad \text{and } d \leq x_0^+ - x_0$$

--that is,

$$x_0^- \leq x_0 - d \quad \text{and } x_0 + d \leq x_0^+.$$

Consequently, for each x,

$$\text{if } x_0 - d < x < x_0 + d \text{ then } x_0^- < x < x_0^+.$$

So, for each $x \in \mathfrak{D}_f$,

$$\text{if } x_0 - d < x < x_0 + d \text{ then } f(x_0) - c < f(x) < f(x_0) + c.$$

In other words,

$$\forall_{x \in \mathfrak{D}_f} [|x - x_0| < d \Rightarrow |f(x) - f(x_0)| < c]$$

--f is continuous at x_0.

The preceeding discussion shows that f is continuous at each x_o such that $a < x_o < b$. To prove that f is continuous, it remains to be shown that f is continuous at a and at b. Since $\mathcal{D}_f = \{x: a \leq x \leq b\}$ and since f is increasing, showing that f is continuous at a amounts to showing that, given $c > 0$, there is a $d > 0$ such that

$$(*)\ \forall_{x \in \mathcal{D}_f} [a \leq x < a + d \Rightarrow f(a) \leq f(x) < f(a) + c].$$

As before, it is sufficient to consider numbers c such that $0 < c \leq f(b) - f(a)$. For such a c, $f(a) < f(a) + c < f(b)$ and, so, there is an argument x_o^+ of f such that $f(x_o^+) = f(a) + c$. It is easy to show that if $d = x_o^+ - x_o$ then $d > 0$ and satisfies (*). The proof that f is continuous at b is carried out in a similar manner. This completes the proof of (b).

Combining (a) and (b), we have the result that

an increasing function whose domain is a segment
has maximal range if and only if it is continuous.

Since, if f is a decreasing function whose domain is a segment, $-f$ is an increasing function whose domain is a segment, and since f is continuous if and only if $-f$ is continuous, and has maximal range if and only if $-f$ does, it follows that the word 'increasing' in our result can be replaced by 'decreasing'--and, hence, by 'monotonic'.

Rather than use (a) to establish our fundamental existence theorem on principal nth roots, it will be more convenient to use the result just obtained to prove an important supplement to Theorem 184 and then use this to establish theorems about the principal nth root functions. The supplement we wish is:

<u>Theorem 187.</u>

Each continuous monotonic function f whose domain is a segment $\overline{a, b}$ has a continuous monotonic inverse of the same type whose domain is the segment $\overline{f(a), f(b)}$.

To prove Theorem 187, we use Theorem 184 and the result we have just established. Suppose that f is a continuous monotonic function whose domain is $\overline{a, b}$. By Theorem 184 [since f is monotonic] f has an inverse,

g, which is also monotonic of the same type. By the if-part of the theorem just established it follows [since, also, f is continuous and its domain is $\overline{a, b}$] that the range of f is the segment $\overline{f(a), f(b)}$. Since this is also the domain of g, it follows that g is a monotonic function whose domain is the segment $\overline{f(a), f(b)}$. Since the range of g is the domain of f, which is the segment $\overline{a, b}$ [and since $a = g(f(a))$, $b = g(f(b))$], it follows that g has maximal range. Consequently, by the only if-part of the theorem just established, g is continuous.

Let's now see how Theorems 184-187 can be applied in the case of the positive-integral power functions, restricted to nonnegative arguments. The nth such function is the function f such that

$$f = \{(x, y), x \geq 0: y = x^n\}.$$

As we have seen [Theorems 185 and 186], this function, whose domain is the ray $\{x: x \geq 0\}$, is continuous and increasing. Consequently, by Theorem 184, f has an inverse g which is also increasing and whose domain is the range of f.

Because f is an increasing function and $0^n = 0$, it follows that the values of f are nonnegative. So, the range of f is a subset of the ray $\{y: y \geq 0\}$. In showing that <u>each</u> nonnegative number belongs to the range of f, as well as that g is continuous, we shall make use of Theorem 187 and of a simple lemma about the function f:

$$\forall_n \forall_{y \geq 0} (y + 1)^n \geq y + 1$$

This lemma can be proved in several ways. One easy proof uses Bernoulli's Inequality [Theorem 162]. According to this, for $a \geq -1$, $(a + 1)^n \geq 1 + na$. Since $n \geq 1$, it follows that, for $a \geq 0$, $na \geq a$. So, for $a \geq 0$, $(a + 1)^n \geq a + 1$. [It is also easy to prove this lemma by induction.]

Now, as to the range of f, suppose that $y_o \geq 0$ and consider the function f_o such that $f_o = \{(x, y), 0 \leq x \leq y_o + 1: y = x^n\}$. [Why '$y_o + 1$'? <u>Hint</u>. Recall the proof on page 9-222 that 5 has a nonnegative 4th root, and see the lemma.] Since f_o is a subset of f, f_o is also a continuous increasing function, and the domain of f_o is the segment $\overline{0, y_o + 1}$. By Theorem 187, f_o has a continuous increasing inverse g_o [a subset of

the inverse g of f] whose domain is the segment $\overline{f_0(0), f_0(y_0 + 1)}$. Since $f_0(0) = 0 \le y_0 < y_0 + 1 \le (y_0 + 1)^n = f_0(y_0 + 1)$, y_0 belongs to the domain of g_0. Since g_0 is a subset of g, $y_0 \in \mathcal{D}_g$.

Now let's prove that g is continuous at y_0. Since the domain of g_0 is $\{y: 0 \le y \le (y_0 + 1)^n\}$ and since, by the lemma, $(y_0 + 1)^n - y_0 \ge 1$, it follows that the domain of g_0 contains all arguments y of g such that $|y - y_0| \le 1$. Since the domain of g_0 contains all arguments of g which are sufficiently close to y_0 and since g_0 is continuous at y_0, so is g.

Consequently, the domain of the inverse g of f is $\{y: y \ge 0\}$, and g is continuous at each of its arguments.

These results on positive-integral power functions justify acceptance of the descriptive definition:

$$\forall_n \forall_{x \ge 0} \sqrt[n]{x} = \text{the } z \ge 0 \text{ such that } z^n = x$$

Our actual course will be to adopt the defining principle for principal roots:

$$\boxed{\text{(PR)} \qquad \forall_n \forall_{x \ge 0} (\sqrt[n]{x} \ge 0 \text{ and } (\sqrt[n]{x})^n = x}$$

The effect of this principle is to introduce the operator '$\sqrt[n]{\ }$' in terms of which we can define the function g of the preceding paragraph--<u>the principal nth root function</u>--as follows:

$$g = \{(x, y), x \ge 0: y = \sqrt[n]{x}\}$$

Since g is the inverse of the function f, we have the theorem:

$$\forall_n \forall_{x \ge 0} \forall_y [(y \ge 0 \text{ and } y^n = x) \Rightarrow y = \sqrt[n]{x}\,] \qquad \text{[Theorem 188]}$$

And, as proved above:

> Each principal positive-integral root function is continuous and increasing on the set of nonnegative numbers. [Theorem 189]

As in the past, we shall follow the custom of abbreviating '$\sqrt[2]{\ }$' to '$\sqrt{\ }$'. Notice, also, that, for each $x \ge 0$, $\sqrt[1]{x} = x$. To prove this, we use Theorem 188:

For $a \ge 0$, $a \ge 0$ and $a^1 = a$. So [by Theorem 188], $a = \sqrt[1]{a}$.

APPENDIX B

[This Appendix fulfills some promises made in section 9.05.]

Irrational numbers.--In section 9.05 the main emphasis was on the rational numbers, but some results were obtained concerning the irrationals. In this Appendix, the main emphasis is on the irrational numbers, but we shall also learn more about rational numbers.

THE IRRATIONALITY OF ROOTS

In section 9.05 we proved that $\sqrt{2}$ is irrational and, from some examples, conjectured:

> Theorem 193.
>
> $\forall_n \forall_m \sqrt[n]{m}$ is irrational unless m is a perfect nth power.

In other words, if a principal root of a positive integer is a rational number then it must be an integer:

$$\forall_n \forall_m [\sqrt[n]{m} \in R \Rightarrow \sqrt[n]{m} \in I]$$

To say that $\sqrt[n]{m} \in R$ is to say that, for some rational number r, $\sqrt[n]{m} = r$--that is, that, for some rational number r, $r \geq 0$ and $r^n = m \in I^+$. So, the theorem last displayed can be restated as:

$$\forall_n \forall_{r \geq 0} [r^n \in I^+ \Rightarrow r \in I]$$

While we are about it, we shall prove the stronger theorem:

$$\forall_n \forall_r [r^n \in I \Rightarrow r \in I]$$

To do so, it is sufficient to prove its contrapositive:

$$\forall_n \forall_r [r \notin I \Rightarrow r^n \notin I]$$

or, more briefly:

$$(*) \qquad \forall_{r \notin I} \forall_n \; r^n \notin I$$

We shall prove (*) by a kind of mathematical induction. What we shall do is to show that, for any given nonintegral rational number r, and any positive integer $m > 1$,

(**) if, for each $p < m$, $r^p \notin I$ then $r^m \notin I$.

When we have shown this, (*) follows easily by the least number theorem [Theorem 108]. For, suppose (*) is not the case--that is, suppose that there is an $r \notin I$ and a positive integer n such that $r^n \in I$. Then, by the least number theorem, there is, for this r, a least such positive integer --say m. By the choice of m, $r^m \in I$ and, for each $p < m$, $r^p \notin I$. Now, since $r \notin I$ and $r^m \in I$, it follows that $m \neq 1$--hence, $m > 1$. Since $m > 1$ and, for each $p < m$, $r^p \notin I$, it follows from (**) that $r^m \notin I$. But, by the choice of m, $r^m \in I$. Hence, the assumption that (*) is not the case leads to a contradiction. Consequently, (*).

Since, as we have seen, Theorem 193 is a consequence of (*), all that remains to complete the proof of Theorem 193 is to complete the proof of (*) by proving (**).

Suppose, then, that $r \notin I$ and that $m > 1$, and suppose that, for each $p < m$, $r^p \notin I$. Since R is closed with respect to multiplication [and $r \in R$], it is clear that, for each p, $r^p \in R$. So, by (R), for each p, there is a positive integer q_p such that $r^p \cdot q_p \in I$. Since I^+ is closed with respect to multiplication, the product of the numbers q_p is a positive integer--say q--which has each of the numbers q_p as a factor. Consequently, q is a positive integer such that, for each $p < m$, $r^p \cdot q \in I$. Hence, by the least number theorem, there is a least such positive integer--that is, a positive integer q_o such that

(1) $$\forall_{p < m}\, r^p \cdot q_o \in I$$

and

(2) $$\forall_{q < q_o} \exists_{p < m}\, r^p \cdot q \notin I.$$

Now, since $m - 1 < m$ and $m > 1$, $m - 1$ is a positive integer less than m and, so, by hypothesis, $r^{m-1} \notin I$. Hence, there is an integer k such that

$$k < r^{m-1} < k + 1. \qquad [k = [\![r^{m-1}]\!]\,]$$

Consequently, $$kq_o < r^{m-1}q_o < kq_o + q_o$$

and, so, $$0 < r^{m-1}q_o - kq_o < q_o.$$

By (1), $r^{m-1}q_o \in I$ and, so, by the closure properties of I, it follows that $r^{m-1}q_o - kq_o \in I$. From the inequations just established it now follows that $r^{m-1}q_o - kq_o$ is a positive integer less than q_o. Hence, by (2), there is a positive integer less than m--say, p--such that

$$r^p(r^{m-1}q_o - kq_o) \notin I.$$

Now, $r^p(r^{m-1}q_o - kq_o) = r^m(r^{p-1}q_o) - k(r^pq_o)$. Hence, for some $p < m$,

$$r^m(r^{p-1}q_o) - k(r^pq_o) \notin I.$$

But, by (1) [and the closure of I with respect to multiplication], $k(r^pq_o) \in I$.

So, $$r^m(r^{p-1}q_o) \notin I.$$ [Explain.]

However, if $p = 1$ then $r^{p-1}q_o = q_o \in I$ and if $1 < p < m$ then, by (1), $r^{p-1}q_o \in I$. So, if $r^m \in I$ then $r^m(r^{p-1}q_o) \in I$. Since this is not the case, $r^m \notin I$.

Consequently, for $r \notin I$ and $m > 1$,

(**) if, for each $p < m$, $r^p \notin I$ then $r^m \notin I$.

INFINITE SETS

The proof in section 9.05 that $\sqrt{2}$ is irrational showed, in particular, that there is at least one irrational number. Theorem 193 goes considerably further by showing how to find lots of irrational numbers. Our purpose, now, is to go still further and show that there are more irrational numbers than there are rational numbers. In fact, we shall prove that there are more irrational numbers between 0 and 1 than there are rational numbers altogether.

To make sense of what has been said we must first discover how to tell when one set has a greater number of members than another set. For a start let's recall the counting principle (C_1) of Unit 8:

> Two sets have the <u>same</u> number of members if and only if the members of one set can be matched in a one-to-one way with those of the other.

This suggests that we agree that a first set has a <u>greater</u> number of members than a second set if, when you match members of the first set in a one-to-one way with all those of the second, you end up having

members of the first set left over. For example, if students enter a classroom and begin sitting, one in each seat, and when the seats are all filled there are students still standing, then there is a greater number of students than of seats.

This seems simple enough. However, in the example just given, the two sets [students and seats] were both finite sets. For each set, you could [if you were in the classroom] give a positive integer [or 0] as the number of its members. The set of irrational numbers and the set of rational numbers are not finite sets--for short, are infinite sets--and, as you will see, infinite sets are tricky. Notions you have formed by thinking about finite sets may not work when you try to apply them to infinite sets.

For example, consider the finite set $\{1, 2, 3, 4, 5, 6\}$ and its subset $\{2, 4, 6\}$. There are many ways of matching some of the members of the first set, one-to-one, with the members of the second set. For example, we might match each of the numbers 2, 4, and 6 with itself. Or, we might match 1 with 2, 3 with 4, and 5 with 6; or 1 with 2, 2 with 4, and 3 with 6. However we do the matching, we find that there are 3 members of the first set left over.

Now, for contrast, consider the set I^+ of all the positive integers, and the subset E whose members are the even positive integers. As before, there are many ways of matching some of the members of I^+, one-to-one, with all the members of E. To begin with, we can match each member of E with itself. In this case, the members of I^+ which are left are just the odd positive integers. On the other hand, we can match with each member of E the positive integer which immediately precedes it:

$$(1)\qquad \begin{array}{ccccccccc} 1, & 2, & 3, & 4, & 5, & \ldots, & & 2n-1, & \ldots \\ & 2, & & 4, & & 6, & \ldots, & 2n, & \ldots \end{array}$$

This time it is the even positive integers which are left over, rather than, as in the first matching, the odd ones. This is not very surprising because, as this second matching shows, there are, according to the counting principle (C_1), the same number of even positive integers as there are of odd positive integers. With each of these two ways of matching, there is the same number of members of I^+ left over. Finally, let's

try a third matching. This time, match with each member of E its half:

(2)
$$\begin{array}{l} 1,\ \ 2,\ \ 3,\ \ 4,\ \ 5,\ \ 6,\ \ 7,\ \ \ldots,\ \ n,\ \ \ldots \\ \quad 2,\ \ \quad 4,\ \ \quad 6,\ \ \ldots\ldots\ldots\ldots,\ \ 2n,\ \ \ldots \end{array}$$

Which members of I^+ are left over this time? Since, given any $n \in I^+$, n is matched with the member 2n of E, there is no member of I^+ left over!

We have discovered a characteristic difference between finite sets and infinite sets. Given a finite set [such as {1, 2, 3, 4, 5, 6}], it is impossible to map all of it in a one-to-one way on any of its *proper* subsets. [A proper subset of a set is one which does not contain all the members of the set.] On the other hand, an infinite set [such as I^+] always has a proper subset on which it can be mapped in a one-to-one way--each infinite set has the same number of members as does some one of its proper subsets. [This property of infinite sets is often used as a definition. So, the matching (2) shows that, according to this definition, I^+ is infinite.]

Evidently, we need to revise our idea as to when a first set has a greater number of members than does a second set. According to this idea, I^+ has a greater number of members than E does, because when we match members of I^+ with all those of E [according to the scheme (1)], there are members of I^+ left over. But, according to the same idea I^+ does not have a greater number of members than E does, because when we match members of I^+ with all those of E [according to the scheme (2)], there are no members of I^+ left over.

In view of this situation, we shall adopt a counting principle:

(C_5) A first set has a greater number of members than a second set if and only if members of the first set can be matched in a one-to-one way with all those of the second set and, *no matter how this is done*, there are members of the first set left over.

Notice that to say that a set A has *more* members than one of its subsets, B, is likely to be confusing. It *may* mean that B is a proper subset of A [A has *other* members than those of B]. Or, it may mean

that A has a greater number of members than B does. If A is a finite set, this confusion may not be important [Explain.]. But, if A is infinite then A can contain <u>other members</u> than those contained in a given subset B and still not have <u>a greater number of members</u> than B.

EXERCISES

<u>A</u>. To show that there is a one-to-one matching between the members of a set A and those of a set B it is sufficient to find a function whose domain is A, whose range is B, and which has an inverse. [For example, the matching (2) on page 9-235 between the positive integers and the even positive integers is determined by the function f for which $f(x) = 2x$, for $x \in I^+$--and is also determined by its inverse f^{-1}, for which $f^{-1}(x) = x/2$, for $x \in E$.] Such a function is called a <u>one-to-one mapping of A on B</u> ['on' means the same as 'onto all of'.].

1. Define a one-to-one mapping g of I^+ on the set O of odd positive integers.

2. Define a one-to-one mapping of I^+ on $\{n: n > 1\}$.

3. On the set of nonnegative integers.

4. On the set of negative integers.

<u>B</u>. Recall that if f is a mapping of A on B and g is a mapping of B on C then, by definition, $g \circ f$ is the mapping of A on C for which $[g \circ f](a) = g(f(a))$, for each $a \in A$.

1. Show that if f is a one-to-one mapping of A on B and g is a one-to-one mapping of B on C then $g \circ f$ is a one-to-one mapping of A on C. [<u>Hint</u>. You must show that $g \circ f$ has an inverse--that is, that, for $a_1 \in A$ and $a_2 \in A$, if $g(f(a_1)) = g(f(a_2))$ then $a_1 = a_2$. Since, by hypothesis, g has an inverse, if $g(f(a_1)) = g(f(a_2))$ then . . . .]

2. Suppose that A_1 and A_2 are sets such that $A_1 \cap A_2 = \emptyset$ and that there is a one-to-one mapping h_1 of A_1 on I^+ and a one-to-one mapping h_2 of A_2 on I^+. Show that there is a one-to-one mapping

h of $A_1 \cup A_2$ on I^+. [Hint. There is (see Part A) a one-to-one mapping f of I^+ on E and a one-to-one mapping g of I^+ on O. By Exercise 1, $f \circ h_1$ is a one-to-one mapping of . . . . If $h(x) = [f \circ h_1](x)$, for $x \in A_1$, and $h(x) = [g \circ h_2](x)$, for $x \in A_2$, then, since . . . , h is a mapping of _____ on $E \cup O$. Since . . . , h is one-to-one.]

COUNTABLY INFINITE SETS

A set is said to be countably infinite if and only if there is some one-to-one matching of its members with those of I^+. I^+ and E are examples of countably infinite sets. Give some other examples.

From the discussion in Part A of the preceding exercises, it follows that a set A is countably infinite if and only if there is a one-to-one mapping of A on I^+ and, also, if and only if there is a one-to-one mapping of I^+ on A.

By Exercise 1 of Part B, a set [the set A] which has the same number of members as a countably infinite set [the set B; with $C = I^+$] is also countably infinite. By the same exercise, two sets [A and C] which are countably infinite [$B = I^+$] have the same number of members [Theorem 194].

By Exercise 2 of Part B, if each of two disjoint sets is countably infinite then their union is also countably infinite [Theorem 195]. So, for example, by Exercises 3 and 4 of Part A, the set I of all integers is countably infinite [Theorem 198].

Although we shall not give the proof, it can be shown [Theorem 196] that each infinite subset of a countably infinite set is countably infinite. So, for example, each subset of I^+ either is a finite set or has the same number of members as I^+ does. It also follows that each infinite set whose members can be matched, one-to-one, with some [or all] of the members of a countably infinite set is, itself, countably infinite.

We now have what we need in order to prove [Theorem 198] that the set R of all rational numbers is countably infinite--that is, that the rational numbers can be matched in a one-to-one way with the positive integers. To establish this somewhat surprising result, we shall begin by considering--not the rational numbers--but the fractions whose numerators and denominators are decimal numerals for positive integers.

We shall prove that this set is countably infinite. To see how, imagine the members of this set arranged in a "square".

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\[6pt]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\[6pt]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\[6pt]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\[6pt]
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Although there would be infinitely many fractions in each row and in each column of such a table, each diagonal line sloping up from the left would contain only a finite number of fractions. [In fact, the nth diagonal, counting from the top, would contain n fractions.] This suggests the following matching:

$$(*)\quad \begin{array}{ccccccccccccccccc}
`\frac{1}{1}' & `\frac{1}{2}' & `\frac{2}{1}' & `\frac{1}{3}' & `\frac{2}{2}' & `\frac{3}{1}' & `\frac{1}{4}' & `\frac{2}{3}' & `\frac{3}{2}' & `\frac{4}{1}' & `\frac{1}{5}' & `\frac{2}{4}' & `\frac{3}{3}' & `\frac{4}{2}' & `\frac{5}{1}' & `\frac{1}{6}' & \cdots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \cdots
\end{array}$$

From the figure, it is clear that there is a one-to-one matching, as suggested above, of all the fractions in question with at least some of the positive integers--to each fraction there is assigned a positive integer, and different integers are assigned to different fractions. [Since people do sometimes doubt this and since, in any case, what is "clear from a figure" is not always so, we shall prove this. But, for the moment, let's accept it.] Since there is a one-to-one matching of all the fractions [whose numerator-numbers and denominator-numbers are positive integers] with at least some of the positive integers, and since the set of fractions is an infinite set, it follows that the set of fractions is a countably infinite set.

Now, each positive rational number is listed many times in our sequence of fractions. But, if we match each such number with its first

listing in the sequence, we have a one-to-one matching of all the positive rational numbers with some of the fractions. Since the set of positive rational numbers is an infinite set, and since the set of fractions is a countably infinite set, it follows that the set of positive rational numbers is countably infinite--that is, there is a one-to-one matching of all the positive rational numbers with all the positive integers.

We can extend any one-to-one matching of the positive rational numbers with the positive integers to a one-to-one matching of R with I [Explain.]. So, since I is countably infinite, so is R. Although each positive integer is a rational number, and although there are infinitely many rational numbers which are not integers, there is exactly the same number of rational numbers as there is of positive integers!

Before going on, let's check that the matching (*) described on page 9-238 actually does match all the fractions whose numerator-numbers and denominator-numbers are positive integers in a one-to-one way with some of the positive integers. [Actually, each positive integer is used in the matching, but we don't need to use this fact.]

Consider the fraction whose numerator-number is m and whose denominator-number is n. Inspection of the "square" on page 9-238 shows that this fraction is in the $(m+n-1)$th diagonal and that, counting up from the left, it is the mth fraction in this diagonal. [For example, the fraction '$\frac{4}{3}$' is the 4th term in the 6th diagonal. Since, for each p, the pth diagonal contains p fractions, there are

$$\sum_{p=1}^{m+n-2} p$$

fractions in the diagonals above the fraction in question and, as we have seen, there are $m - 1$ fractions in its own diagonal which precede it. So, the number of fractions which precede it in the suggested matching is

$$\sum_{p=1}^{m+n-2} p + (m - 1)$$

--that is,

$$\frac{(m+n-2)(m+n-1)}{2} + (m - 1). \qquad \text{[Explain.]}$$

Hence, the fraction whose numerator-number is m and whose denominator-number is n is to be matched with the positive integer

$$\frac{(m + n - 2)(m + n - 1)}{2} + m.$$

Our problem is to prove that the matching so-described actually does have the property suggested by the figure--that no two fractions are matched with the same positive integer. This amounts to proving that

(*) if $\frac{(m+n-2)(m+n-1)}{2} + m = \frac{(p+q-2)(p+q-1)}{2} + p$
then $m = p$ and $n = q$.

For simplicity, suppose that $m + n - 1 = j$ and $p + q - 1 = k$. Then, $m \leq j$ and $p \leq k$, and what we want to show is that

if $(j - 1)j + 2m = (k - 1)k + 2p$ then $m = p$ and $n = q$.

To do so, suppose that $(j - 1)j + 2m = (k - 1)k + 2p$. Since $p \leq k$, it follows that $(k - 1)k + 2p \leq (k - 1)k + 2k = (k + 1)k$. Hence, $(j - 1)j + 2m \leq (k + 1)k$, and [since $m > 0$]

(1) $$(k + 1)k > (j - 1)j.$$

Now, the function f for which, for each $i \in I^+$, $f(i) = (i - 1)i$ is an increasing function. [Suppose that $i_2 > i_1$. It follows that $i_2 - 1 > i_1 - 1$ and, since $i_2 > 0$, that $(i_2 - 1)i_2 > (i_1 - 1)i_2$. Also, since $i_2 > i_1$ and $i_1 - 1 \geq 0$, it follows that $(i_1 - 1)i_2 \geq (i_1 - 1)i_1$. Hence, if $i_2 > i_1$, $(i_2 - 1)i_2 > (i_1 - 1)i_1$.] And, (1) says that

$$f(k + 1) > f(j).$$

Since each increasing function has an increasing inverse, it follows that $k + 1 > j$--that is, that $k \geq j$. Similarly [since $m \leq j$], $j \geq k$ and, so, $j = k$.

Since, by hypothesis, $(j - 1)j + 2m = (k - 1)k + 2p$, it follows that $m = p$. And, since $j = m + n - 1$ and $k = p + q - 1$, it follows, because $j = k$, that $m + n = p + q$. So, since $m = p$, $n = q$. Consequently, (*).

Notice that, although we have spoken of fractions and their numerator-numbers and denominator-numbers, we might as well have spoken of ordered pairs of positive integers. So, what we have shown is that the Cartesian product $I^+ \times I^+$ is countably infinite. It is only a small step from this to the result [Theorem 197] that the Cartesian product of any two countably infinite sets is countably infinite.

UNCOUNTABLY INFINITE SETS

You may have begun to wonder whether there are any infinite sets which are not countably infinite. There are, and, as we shall see, the set of irrational numbers is one such set. To prove that this is the case, it is sufficient to prove that the set of all real numbers is not countably infinite. For, as we have proved, the set of rational numbers is countably infinite and, if the set of irrational numbers were also countably infinite, it would follow that the set of real numbers--being the union of two countably infinite sets--was countably infinite. So, once we prove that this is not the case, it will follow that the set of irrational numbers is not countably infinite, either.

In order to prove that the set of real numbers is not countably infinite, we need a result about real numbers which depends on the least upper bound principle. What we need to know is that each "infinite decimal" represents a real number [Theorem 199a], and that different decimals all of whose digits are '0's and '1's represent different numbers [Theorem 199b]. If you accept this, the proof that the set of real numbers is not countably infinite is very simple. Here it is:

> Consider the set of all nonnegative real numbers less than 1 which can be represented by infinite decimals using only the digits '0' and '1'. If the set of all real numbers is countably infinite, then [by Theorem 196] this subset is also countably infinite. We shall show that it is not. To do so, suppose we have any one-to-one matching of some of these "0, 1-numbers" with all the positive integers. The following figure suggests such a matching:
>
> $$\begin{array}{ccc} 1 & \longleftrightarrow & 0.a_{11}a_{12}a_{13}\cdots \\ 2 & \longleftrightarrow & 0.a_{21}a_{22}a_{23}\cdots \\ 3 & \longleftrightarrow & 0.a_{31}a_{32}a_{33}\cdots \\ \vdots & & \vdots \end{array}$$
>
> [Each of the 'a_{mn}' stands for either a '0' or a '1'.] Now, consider the real number which is represented by the decimal whose nth digit after its decimal point is a '0' if $a_{nn} = 1$ and is a '1' if $a_{nn} = 0$. This real number is a 0, 1-number which is

different from all of those which have been matched with positive integers. [For each n, it is different from the number matched with n because the decimal representations of the two numbers have different digits at their nth places.] So, no matter how we match 0, 1-numbers, one-to-one, with the positive integers, there will be some 0, 1-number left over--that is, there is no one-to-one matching of all the 0, 1-numbers with the positive integers. Hence, the set of 0, 1-numbers is not countably infinite and, even more so, the set of all real numbers is not countably infinite.

Notice that we have also shown [Theorem 200a] that the set of real numbers in the segment $\overleftrightarrow{0, 1}$ is not countably infinite [Explain.]. So, since the set of rational numbers in the segment is countably infinite [Why?], it follows that the set of irrational numbers between 0 and 1 is not countably infinite. We can now prove, as we promised, that there is a greater number of irrational numbers between 0 and 1 than there are rational numbers altogether [Theorem 200b]. What this means, you recall, is that there are enough irrational numbers between 0 and 1 to match, one-to-one, with all the rational numbers, but that, <u>however this is done</u>, there will be some irrational numbers left over. Now, there is at least one irrational number between 0 and 1. [One such number is $\sqrt{2}/2$. Another can be found by enumerating the <u>rational</u> 0, 1-numbers which are nonnegative and less than 1 and applying the diagonal procedure" used, above, in the proof that the set of real numbers is not countably infinite.] If a is such an irrational number then, for each n, a/n is another [Explain.] In fact, the matching:

$$n \longleftrightarrow a/n$$

is a one-to-one matching of some of the irrational numbers between 0 and 1 with all the positive integers. So, since there is a one-to-one matching of all the positive integers with all the rational numbers, there are enough irrational numbers between 0 and 1 to match, one-to-one, with all the rational numbers. But, since the set of rational numbers is countably infinite and the set of irrational numbers between 0 and 1 is not, however such a matching is carried out, there will be irrational numbers left over.

To put the results just obtained on a firm basis, we need to know that each decimal of the form:

$$(*) \qquad 0.a_1a_2a_3\ldots,$$

where each 'a_m' stands for either a '0' or a '1', "represents" a nonnegative real number less than 1, and that two such decimals [which differ in some decimal place] represent different real numbers. To begin with, we need to know what is meant by saying that a decimal represents a real number. What real number?

In the following discussion, we shall, for completeness, consider decimals of the form (*) in which each 'a_m' may be any of the ten digits '0', '1', ..., '9'. The meaning of a repeating decimal has already been discussed in Unit 8 in connection with sums of infinite geometric progressions. Also, that each nonnegative real number less than 1 can be represented by a decimal like (*) [and what this means] follows from an optional section in Unit 8 on base-m approximations of real numbers. The following discussion is, however, independent of this material from Unit 8.

Given any decimal (*), each of whose digits is '0', '1', ..., or '9', consider the sequence whose mth term is the number $a_m \cdot 10^{-m}$. For this sequence, consider its continued sum sequence s, whose nth term s_n, is

$$\sum_{p=1}^{n} a_p \cdot 10^{-p}.$$

Since $a_1 \geq 0$, $s_1 \geq 0$. Since, for each p, $a_p \leq 9$, so that $a_p \cdot 10^{-p} \leq 9 \cdot 10^{-p}$, it follows [see Theorem 143] that, for each n,

$$s_n \leq \sum_{p=1}^{n} 9 \cdot 10^{-p}.$$

By Theorem 167c on the continued sum sequence of a geometric progression, it follows that, for each n,

$$\sum_{p=1}^{n} 9 \cdot 10^{-p} = \frac{9 \cdot 10^{-1}(1 - 10^{-n})}{1 - 10^{-1}}$$

$$= 1 - 10^{-n} < 1.$$

Hence, for each n, $s_n < 1$. Consequently, the set S of those real numbers which occur as terms of the sequence s is a [nonempty] set of real numbers which has an upper bound. It follows from the ℓubp that S has a least upper bound--say, ℓ. Since $s_1 \in S$ and $s_1 \geq 0$, $\ell \geq 0$. Since 1 is an upper bound of S, $\ell \leq 1$. This number ℓ which is uniquely determined by the decimal (*) is, by definition, the number represented by (*). As we have shown, it belongs to the segment $\overline{0, 1}$. [This, and the work in Unit 8 on base-m approximations completes the proof of Theorem 199a.]

You are accustomed to getting rational approximations for real numbers by "rounding-off" their decimal representations. For example, the decimal representation of $\sqrt{2}/2$ begins:

$$0.7071\ldots$$

So, you say:

$$\sqrt{2}/2 \doteq 0.7071$$

--that is, you take the number s_4 as an approximation to ℓ. Let's see how this works in general. If $m \geq n$ then

$$\begin{aligned} s_m &= \sum_{p=1}^{m} a_p \cdot 10^{-p} \\ &= \sum_{p=1}^{n} a_p \cdot 10^{-p} + \sum_{p=n+1}^{m} a_p \cdot 10^{-p} \\ &= s_n + \sum_{p=n+1}^{m} a_p \cdot 10^{-p} \end{aligned}$$

Now, using this result, we can show [just as we showed that, for each n, $s_n < 1$] that, for each $m \geq n$,

(1) $\qquad s_m - s_n < 10^{-n}$. $\qquad$ [Use Theorems 143 and 167c.]

Since for each p, $a_p \geq 0$, it follows, also, that

(2) $\qquad s_m - s_n \geq 0$.

Consequently [(2)], for each $m < n$ and [(1)] for each $m \geq n$, $s_m < s_n + 10^{-n}$. It follows that $s_n + 10^{-n}$ is an upper bound of the set S

whose least upper bound is the number ℓ which is represented by (*). Hence, for any n,

$$\ell \leq s_n + 10^{-n}.$$

So, since $s_n \in S$ and ℓ is an upper bound of S, we have, for each n,

$$(3) \qquad s_n \leq \ell \leq s_n + 10^{-n}.$$

Applying this to the case of $\sqrt{2}/2$ we see [n = 4] that:

$$0.7071 \leq \frac{\sqrt{2}}{2} \leq 0.7071 + 10^{-4} \qquad [= 0.7072]$$

[Since $\sqrt{2}/2$ is not rational the '$\leq$'s can be replaced by '$<$'s.]

So far, we have shown that each decimal of the form (*) represents a real number in the segment $\overleftrightarrow{0, 1}$. [And, in the optional section of Unit 8 previously referred to, we proved that each real number in $\overline{0, 1}$ is represented by a decimal of the form (*).] The next question is: Do different decimals represent different numbers? As you know, the answer is 'No'--the repeating decimals '$0.1\bar{0}$' and '$0.0\bar{9}$' both represent the number 1/10. We shall see that this is the only kind of situation in which two decimals do represent the same number. More explicitly, we shall see that if (*) represents ℓ, and a different decimal:

$$(*') \qquad 0.a_1' a_2' a_3' \ldots ,$$

represents ℓ', then, $\ell = \ell'$ if and only if, when the first digit in which (*) and (*') differ is the nth

$$(a) \qquad a_n = a_n' + 1 \qquad [\text{or } a_n' = a_n + 1]$$

and (b) $\qquad$ for $p > n$, $a_p = 0$ and $a_p' = 9$ $\quad$ [or $a_p = 9$ and $a_p' = 0$].

[Incidentally, it follows from this that two decimals each of whose digits is either a '0' or a '1' must represent different numbers.]

To prove that this is the case, suppose that (*) and (*') are different decimals which represent the numbers ℓ and ℓ', respectively, and that the first digit in which (*) and (*') differ is the nth. Since $a_n \neq a_n'$, we may assume "without loss of generality" that $a_n > a_n'$. Since, for $p < n$, $a_p = a_p'$, it follows [Theorem 135] that, for each $m \geq n$,

$$s_m - s_m' = \sum_{p=n}^{m} a_p \cdot 10^{-p} - \sum_{p=n}^{m} a_p' \cdot 10^{-p}$$

and, so [Theorem 136, etc.]

$$(4) \qquad s_m - s_m' = (a_n - a_n')10^{-n} + \sum_{p=n+1}^{m} (a_p - a_p')10^{-p}.$$

There are two cases to consider: $a_n - a_n' > 1$ and: $a_n - a_n' = 1$

In the first case, $a_n - a_n' \geq 2$ and, by (4) [$m = n$],

$$s_n - s_n' \geq 2 \cdot 10^{-n}.$$

Now, by (3), since (*) represents the number ℓ,

$$s_n \leq \ell$$

and, since (*′) represents the number ℓ',

$$\ell' \leq s_n' + 10^{-n}.$$

From these it follows that

$$s_n - s_n' - 10^{-n} \leq \ell - \ell'$$

and, since $s_n - s_n' - 10^{-n} \geq 2 \cdot 10^{-n} - 10^{-n}$, that $\ell - \ell' \geq 10^{-n}$. Hence [in this first case], $\ell \neq \ell'$.

Now, consider the second case, that in which $a_n - a_n' = 1$. By (4), for each $m \geq n$,

$$(5) \qquad s_m - s_m' = 10^{-n} + \sum_{p=n+1}^{m} (a_p - a_p')10^{-p}.$$

We shall consider two subcases: for each $p > n$, $a_p = 0$ and $a_p' = 9$, and: for some integer--say q--$q > n$, and ($a_q > 0$ or $a_q' < 9$).

In the first subcase, for each $p > n$, $a_p - a_p' = -9$. Consequently, for each $m \geq n$,

$$\sum_{p=n+1}^{m} (a_p - a_p')10^{-p} = \sum_{p=n+1}^{m} -9 \cdot 10^{-p}$$

$$= \frac{-9 \cdot 10^{-(n+1)}\left(1 - 10^{n-m}\right)}{1 - 10^{-1}}$$

$$= -10^{-n}\left(1 - 10^{n-m}\right)$$

$$= -10^{-n} + 10^{-m}.$$

Hence [from (5)], for each $m \geq n$,

$$s_m - s_m' = 10^{-n} - 10^{-n} + 10^{-m} = 10^{-m}.$$

Now, by (3), since (*) represents ℓ and (*') represents ℓ',

$$s_m \leq \ell \leq s_m + 10^{-m}$$

and

$$s_m' \leq \ell' \leq s_m' + 10^{-m}.$$

From these it follows that

$$s_m - s_m' - 10^{-m} \leq \ell - \ell' \leq s_m + 10^{-m} - s_m'.$$

Since $s_m - s_m' = 10^{-m}$, it follows that, for each $m \geq n$,

$$(6) \qquad 0 \leq \ell - \ell' \leq 2 \cdot 10^{-m}.$$

From this last result it follows that $\ell = \ell'$. For, if not, $\ell - \ell' > 0$ and, by Theorem 165,

$$(1/10)^m < \frac{\ell - \ell'}{2}$$

if $m > \dfrac{1}{\frac{\ell - \ell'}{2}(1 - \frac{1}{10})} = \dfrac{20}{9(\ell - \ell')}$. Now, there is a positive integer m which is greater than or equal to both n and $\dfrac{20}{9(\ell - \ell')}$. Hence, if $\ell - \ell' > 0$, it follows that, for such an m, both

$$\ell - \ell' \leq 2 \cdot 10^{-m} \qquad \text{and} \qquad \ell - \ell' > 2 \cdot 10^{-m}.$$

This being impossible, $\ell - \ell' \not> 0$ and, by (6), $\ell = \ell'$.

Finally, consider the second subcase--that in which, for a certain integer q, $q > n$ and ($a_q > 0$ or $a_q' < 9$). Since this is a subcase of the second case we have, by (5), that, for $m \geq n$,

$$s_m - s_m' = 10^{-n} + \sum_{p=n+1}^{m} (a_p - a_p')10^{-p}.$$

Since, in any case, $0 \leq a_p \leq 9$ and $0 \leq a_p' \leq 9$, it follows that, for $p > n$, $a_p - a_p' \geq -9$. Moreover, $q > n$ and since, by hypothesis, $a_q - a_q' > -9$, $a_q - a_q' \geq -9 + 1$. This being the case, it follows that, for $m \geq q$,

$$\sum_{p=n+1}^{m} (a_p - a_p')10^{-p} \geq \sum_{p=n+1}^{m} -9 \cdot 10^{-p} + 1 \cdot 10^{-q}$$

$$= -10^{-n} + 10^{-m} + 10^{-q}.$$

Hence, for $m \geq q$,

$$s_m - s_m' \geq 10^{-n} - 10^{-n} + 10^{-m} + 10^{-q}$$
$$= 10^{-m} + 10^{-q}.$$

Now, just as in the first case, it follows from (3) that

$$s_m - s_m' - 10^{-m} \leq \ell - \ell'$$

and, since $s_m - s_m' - 10^{-m} \geq 10^{-q}$, it follows that $\ell - \ell' \geq 10^{-q}$. Hence, $\ell \neq \ell'$.

Combining our results we see, from the first subcase of the second case, that if (*) and (*′) satisfy (a) and (b) then $\ell = \ell'$, while if (a) is not satisfied [first case], or (a) is satisfied but (b) is not [second subcase of second case], then $\ell \neq \ell'$. Consequently, $\ell = \ell'$ if and only if (*) and (*′) satisfy both (a) and (b). [This completes the proof of Theorem 199b.]

EXERCISES

1. Prove that there are exactly as many real numbers between any two given real numbers as there are between 0 and 1. [Hint. Suppose that a and b are real numbers, with $a < b$. Establish a one-to-one mapping between $\overline{0, 1}$ and $\overline{a, b}$ by defining the simplest increasing function f you can think of for which $f(0) = a$ and $f(1) = b$.]

2. Prove that there are exactly as many

   (a) rational numbers (b) irrational numbers

   between any two given rational numbers as there are between 0 and 1. [Hint. Show that, if a and b are rational, the function f of Exercise 1 matches rational numbers with rational numbers and irrational numbers with irrational numbers.] [A similar result holds if the given numbers are not both rational. But the proof, at least of (b), is more difficult.]

3. Prove that between each two rational numbers there is an irrational number.

THE DENSITY OF THE RATIONALS IN THE REALS

In section 9.05, you proved that the real numbers are dense--between any two real numbers there is a real number. In addition to this, you proved that between any two rational numbers there is a rational number. Also, in solving Exercise 3, above, you have proved that between any two rational numbers there is an irrational number--in fact, as follows from Exercise 2, between any two rational numbers there are uncountably many irrational numbers, and a countable infinity of rational numbers. By now, these results should not be too surprising. However, there is still one more result which may surprise you--in spite of the fact that there is a much larger number of irrational numbers than of rational numbers, there is a rational number between any two real numbers:

$$\forall_x \forall_{y>x} \exists_r \; x < r < y \qquad \text{[Theorem 201]}$$

--the rational numbers are dense in the reals.

To see how to prove that this is the case, suppose that you are given real numbers a and b with b > a. A little thought suggests that if we could find a positive integer q such that there is an integer--say, k--between qa and qb then k/q would be a rational number between a and b.

Now, for any q, $[\![qa]\!] + 1$ is an integer greater than qa and, since $[\![qa]\!] \le qa$, $[\![qa]\!] + 1 \le qa + 1$. So, we are

all set if we can find an integer q such that $qa + 1 < qb$. But, since $b - a > 0$, this means, merely, that $q > 1/(b - a)$, and the cofinality principle tells us that there is such a q. Summing up--for $b > a$, $([\![qa]\!] + 1)/q$ is a rational number between a and b, for any $q > 1/(b - a)$ [and, by the cofinality principle, there is such an integer q].

EXERCISES

1. Prove: $\forall_x \forall_{y>x} \exists_r \exists_s \; x < r < s < y$

2. Prove that between any two real numbers there is
   (a) a countable infinity of rational numbers, and
   (b) an uncountable infinity of irrational numbers.

APPENDIX C

[The purpose of this Appendix is to provide justification for clause (4_1) of the definition, on page 9-92, of the exponential functions, and to furnish proofs of Theorems 202-211 and of Theorem 220. The reader should previously have studied Appendix A.]

<u>The exponential functions</u>.--For any $a > 0$, the exponential function with base a and rational arguments has been defined in section 9.06:

$$(3) \qquad \forall_r \forall_{m,\ rm \in I}\ a^r = (\sqrt[m]{a})^{rm}$$

Using this definition we have proved Theorems 203-211, as stated on page 9-75, , under the restriction that the exponents are rational numbers. [In this Appendix, we shall refer to these restricted theorems as Theorems 203_r-211_r.]

In section 9.07 the definition of the exponential functions was completed by adopting:

$$(4_1) \quad \forall_{x>1} \forall_u\ x^u = \text{the least upper bound of } \{y: \exists_{r<u}\ y = x^r\}$$

$$(4_2) \quad \forall_{0<x<1} \forall_u\ x^u = (1/x)^{-u}$$

$$(4_3) \quad \forall_u\ 1^u = 1 \text{ and } \forall_{u>0}\ 0^u = 0 \ [\text{and, as previously, } 0^0 = 1]$$

As remarked on page 9-92, these definitions, particularly (4_1), need some justification. As a basis for this justification, and for the proof of Theorem 202, we need to establish some further properties of the exponential functions with rational arguments. [As we shall see, Theorems 203-211 follow easily from Theorem 202 and Theorems 203_r-211_r.] Theorem 202 is:

> For each $x > 0$, the exponential function with base x is a continuous function whose domain is the set of all real numbers; if $x \neq 1$, its range is the set of all positive numbers; it is decreasing if $0 < x < 1$ and increasing if $x > 1$.

The additional properties we need of the exponential functions with rational arguments are like those which are asserted in Theorem 202 to

hold of the complete exponential functions:

> For $0 < a \neq 1$, the exponential function with base a and rational arguments is a continuous monotonic function--decreasing if $0 < a < 1$ and increasing if $a > 1$.
>
> For $0 < a < 1$, given $M > 0$, there is an N such that, for each $r > N$, $a^{-r} > M$ and $0 < a^r < 1/M$.
>
> For $a > 1$, given $M > 0$, there is an N such that, for each $r > N$, $a^r > M$ and $0 < a^{-r} < 1/M$.

[The restriction '$a \neq 1$' is introduced, above, to simplify the statement which needs to be proved. No such restriction is included in Theorem 202 because such a restriction would complicate the proofs of Theorems 203-211. However, the assertions made about the exponential function with base 1 in Theorem 202 are obviously correct. By (4_3), this function is a constant function and is defined for all real numbers--and each constant function is continuous. So, in proving Theorem 202 we need treat only two cases--the case in which the base is between 0 and 1 and that in which the base is greater than 1. Incidentally, as far as the consistency of (4_3) is concerned, it is enough to check that it is consistent with (3). This is the case. By (3), for each r, $1^r = (\sqrt[m]{1})^{rm}$, for any m such that $rm \in I$ and, by previous theorems [Theorems 190c and 150a], $(\sqrt[m]{1})^{rm} = 1$.]

We proceed, first, to establish the properties listed above of the exponential functions with rational arguments. Next, we shall use these results to justify (4_1) and (4_2). Third, we shall prove Theorem 202. Then, we shall prove Theorems 203-211. Finally we shall prove a theorem--Theorem 220--on power functions--the functions

$$\{(x, y), x > 0: y = x^u\}, \text{ for some } u.$$

MORE ON RATIONAL EXPONENTS

We begin by proving that, for $0 < a \neq 1$, the exponential function with base a and rational arguments is monotonic--decreasing if $0 < a < 1$ and increasing if $a > 1$. The proof is similar to that outlined for $a = 2$, in Part E on pages 9-87 and 9-88.

For $0 < a \neq 1$, suppose that $r_2 > r_1$ and consider $a^{r_2} - a^{r_1}$. Since R is closed with respect to subtraction, it follows by Theorem 205, that, for any r and s, $a^r = a^{r-s} \cdot a^s$. So,

$$(*) \qquad a^{r_2} - a^{r_1} = (a^{r_2 - r_1} - 1)a^{r_1}.$$

Since, by Theorem 203_r, $a^{r_1} > 0$, it follows that a^{r_2} is less than or greater than a^{r_1} according as $a^{r_2 - r_1}$ is less than or greater than 1. We wish to show [assuming $r_2 > r_1$] that this is the case according as $0 < a < 1$ or $a > 1$. More conveniently stated, we wish to show, for $a > 0$, that, if $r > 0$,

$$a^r < 1^r \text{ or } a^r > 1^r \text{ according as } a < 1 \text{ or } a > 1.$$

Now, by (3) [and (R)], there is an m such that $rm \in I$ and

$$a^r = (\sqrt[m]{a})^{rm} \text{ and } 1^r = (\sqrt[m]{1})^{rm}.$$

Moreover, if $r > 0$ then $rm > 0$ and, so $rm \in I^+$. Consequently, the (rm)th power function restricted to positive arguments is increasing [Theorem 185] and, so, a^r is less than or greater than 1^r according as $\sqrt[m]{a}$ is less than or greater than $\sqrt[m]{1}$. But, since the mth principal root function is increasing [Theorem 189], this is the case according as a is less than or greater than 1.

Our next job is to show that, for $0 < a \neq 1$, the exponential function with base a and rational arguments is <u>continuous</u>. We treat, first, the case in which $a > 1$. What we wish to prove is that, for $a > 1$ and any r_o, given $c > 0$, there is a $d > 0$ such that

$$(**) \qquad \forall_r \,[\,|r - r_o| < d \Rightarrow |a^r - a^{r_o}| < c].$$

Taking a tip from (*)[in the proof of monotonicity], we begin by considering the case in which $r_o = 0$, in which case (**) reduces to:

$$\forall_r \,[\,|r| < d \Rightarrow |a^r - 1| < c]$$

Since, for $a > 1$, the exponential function with base a and rational arguments is increasing, we know that

$$\text{if } -\frac{1}{n} < r < \frac{1}{n} \text{ then } a^{-1/n} < a^r < a^{1/n}.$$

So, if, given $c > 0$, we can find an n such that both $a^{-1/n}$ and $a^{1/n}$ differ from 1 by at most c, it will follow that, for this n,

$$\text{if } |r| < \frac{1}{n} \text{ then } a^r \text{ differs from 1 by less than } c$$

--and, for this c, we can choose $d = 1/n$. We proceed, then, to estimate $a^{1/n}$--that is, $\sqrt[n]{a}$--and its reciprocal. To do so, we shall use Bernoulli's Inequality [Theorem 162].

Since the principal nth root function is increasing, it follows that, for $a > 1$, $\sqrt[n]{a} > \sqrt[n]{1} = 1$. So, if $h = \sqrt[n]{a} - 1$ then $h > 0$. Also, $a = (1+h)^n$ and, by Bernoulli's Inequality, $a \geq 1 + nh$ and, so, $h \leq \frac{a-1}{n}$. Consequently, for $a > 1$,

$$(1) \qquad 0 < \sqrt[n]{a} - 1 \leq \frac{a-1}{n}.$$

Furthermore,

$$1 - \frac{1}{\sqrt[n]{a}} = \frac{\sqrt[n]{a} - 1}{\sqrt[n]{a}}$$

and, since, for $a > 1$, $1/\sqrt[n]{a} < 1$,

$$\frac{\sqrt[n]{a} - 1}{\sqrt[n]{a}} < \frac{\sqrt[n]{a} - 1}{1} = \sqrt[n]{a} - 1.$$

Consequently, for $a > 1$,

$$(2) \qquad 0 < 1 - \frac{1}{\sqrt[n]{a}} < \frac{a-1}{n}.$$

From (1) and (2) we see that, for $a > 1$, both $a^{-1/n}$ and $a^{1/n}$ differ from 1 by at most $\frac{a-1}{n}$. Consequently, if we choose n so that $\frac{a-1}{n} \leq c$ --that is, for $c > 0$, so that $n \geq (a-1)/c$--it will follow that both $a^{-1/n}$ and $a^{1/n}$ differ from 1 by at most c. In fact, for such an n,

$$-c < a^{-1/n} - 1 < 0 < a^{1/n} - 1 \leq c.$$

Since, as noted previously,

$$\text{if } -\frac{1}{n} < r < \frac{1}{n} \text{ then } a^{-1/n} - 1 < a^r - 1 < a^{1/n} - 1,$$

it follows that, for $n \geq (a-1)/c$,

$$\text{if } |r| < \frac{1}{n} \text{ then } |a^r - 1| < c.$$

Since, by the cofinality principle, there is an n such that $n \geq (a-1)/c$, it follows that, given $c > 0$, there is a $d > 0$ such that

$$\forall_r \, [\, |r| < d \Rightarrow |a^r - 1| < c]$$

--for $a > 1$, the exponential function with base a and rational arguments is continuous at 0.

It is now easy to translate this result from 0 to any argument r_o. For, as we know,

$$a^r - a^{r_o} = (a^{r - r_o} - 1)a^{r_o}$$

and, so, since $a^{r_o} > 0$ [Theorem 203],

$$|a^r - a^{r_o}| = |a^{r - r_o} - 1|a^{r_o}.$$

Hence, if we wish, for some $c > 0$, to restrict r so that $|a^r - a^{r_o}| < c$, all we need do is restrict $r - r_o$ so that $|a^{r - r_o} - 1| < c/a^{r_o}$. Since $c/a^{r_o} > 0$, this can, as we have seen above, be done by requiring that $|r - r_o| < \frac{1}{n}$ where n is some positive integer not less than $\frac{a - 1}{c/a^{r_o}}$. [The least such positive integer is $-\left[\!\left[\frac{(1 - a)a^{r_o}}{c}\right]\!\right]$.]

Consequently, for $a > 1$, the exponential function with base a and rational arguments is continuous at each of its arguments.

To treat the case in which $0 < a < 1$ we note that, in this case, $1/a > 1$ [Theorem 164] and that, by Theorem 210_r, for each r, $a^r = (1/a)^{-r}$ Consequently, for each r and r_o, $|a^r - a^{r_o}| = |(1/a)^{-r} - (1/a)^{-r_o}|$ and, since $1/a > 1$ [and R is closed with respect to oppositing] it follows from the result just proved that, given r_o and $c > 0$, there is a $d > 0$ such that

$$\text{if } |-r - -r_o| < d \text{ then } |a^r - a^{r_o}| < c.$$

Since $-r - -r_o = -(r - r_o)$, this is equivalent to:

$$\text{if } |r - r_o| < d \text{ then } |a^r - a^{r_o}| < c.$$

Hence, for $0 < a < 1$, the exponential function with base a and rational arguments is continuous at each of its arguments.

* * *

Remark. When we come to apply the results just obtained to prove Theorem 202, we shall need to take note of a slight refinement which can be made in the preceding proof of continuity in the case $a > 1$. A sketch of a graph of an exponential function with base greater than 1 will convince you that the graph is less steep for small arguments than for large ones. One consequence of this is that once we have found, for

some argument r_1 and some $c > 0$, a $d > 0$ such that

$$\forall_r [\, |r - r_1| < d \Rightarrow |a^r - a^{r_1}| < c],$$

we can be sure that, for each $r_0 \leq r_1$,

$$\forall_r [\, |r - r_0| < d \Rightarrow |a^r - a^{r_0}| < c].$$

In short, given $c > 0$, a $d > 0$ which "works" for some argument will also work for each smaller argument. In particular, for $a > 1$, given an r_1, there is a $d > 0$ which will work for each argument $r_0 \leq r_1$ -- that is, a $d > 0$ such that

$$(***)\ \forall_{r_0 \leq r_1} \forall_r [\, |r - r_0| < d \Rightarrow |a^r - a^{r_0}| < c].$$

To prove this last [which is what we need for the proof of Theorem 202] it is sufficient to exhibit such a number d. In view of the preceding proof, this is easy. Let N_1 be some integer greater than $\frac{(a - 1)a^{r_1}}{c}$ and let $d = 1/N_1$. From the proof we know that this d works for r_1. Suppose that $r_0 < r_1$. Then since $a > 1$, $a^{r_0} < a^{r_1}$ and, consequently $\frac{(a - 1)a^{r_0}}{c} < \frac{(a - 1)a^{r_1}}{c}$. Hence, N_1 is greater than $\frac{(a - 1)a^{r_0}}{c}$ and, as shown in the proof, it follows that $1/N_1$ works for r_0. Consequently, $1/N_1$ works, as a number d, for the given $c > 0$ and each $r_0 \leq r_1$.

[In technical terms, what we have just shown is that, for $a > 1$ and any r_1, the exponential function with base a and rational arguments is uniformly continuous on $\{r: r \leq r_1\}$.]

* * *

Finally, concerning exponential functions with rational arguments, we need to know that

for $0 < a < 1$, given $M > 0$, there is an N such that

$$\forall_{r > N}(a^{-r} > M \text{ and } 0 < a^r < 1/M)$$

and that

for $a > 1$, given $M > 0$, there is an N such that

$$\forall_{r > N}(a^r > M \text{ and } 0 < a^{-r} < 1/M).$$

The essential content of these statements is that, for $0 < a \neq 1$, the exponential function with base a and rational arguments has, among its values, both arbitrarily large numbers and arbitrarily small positive numbers.

It will be sufficient to prove the second statement. [The first follows from the second somewhat as continuity for base $0 < a < 1$ follows from continuity for base $a > 1$.]

For $a > 1$, $a = 1 + h$, with $h > 0$. So, by Bernoulli's Inequality, for each $k \geq 0$, $a^k \geq 1 + kh$. Now, given any number $M > 0$ [no matter how large], $1+kh \geq M$ for each $k \geq (M - 1)/h$. If N is such an integer k--say, the least one--then, since, for $a > 1$, the exponential function with base a and rational arguments is increasing, it follows that if $r > N$ then $a^r > a^N \geq M$. Moreover, since $a^r > 0$, $M > 0$, and $1/a^r = a^{-r}$, it follows that if $r > N$ then $a^{-r} < 1/M$. And, of course, $a^{-r} > 0$.

JUSTIFICATION OF (4_1) AND (4_2)

To justify:

$$(4_1)\ \forall_{x > 1} \forall_u\, x^u = \text{the least upper bound of } \{y: \exists_{r < u}\, y = x^r\}$$

we must show, first, that for $a > 1$ and any b,

$$\{y: \exists_{r < b}\, y = a^r\}$$

has a least upper bound. Second, to prove that (4_1) is consistent with earlier definitions, we must prove that, for $a > 1$ and any r_1,

the least upper bound of $\{y: \exists_{r < r_1}\, y = a^r\}$ is,
if $r_1 p \in I$, $(\sqrt[p]{a})^{r_1 p}$.

By the least upper bound principle, we shall have done the first job once we have shown that, for $a > 1$ and any b, $\{y: \exists_{r < b}\, y = a^r\}$ is nonempty and has some upper bound. Now, by Theorem 201 [density of the rationals in the reals] [or directly from the cofinality principle], it follows that, given b, there is a rational number r_0 such that $r_0 < b$ and a rational number r_1 such that $b < r_1$. Since $r_0 < b$, a^{r_0} belongs to the set in question--so, the set is nonempty. On the other hand, since $b < r_1$, it follows that if $r < b$ then $r < r_1$. Since, for $a > 1$, the exponential function with base a and rational exponents is increasing, it follows that, if $r < b$ then $a^r < a^{r_1}$--so, the set in question has a^{r_1} as an upper bound.

Now, we tackle the second job. Much as in the preceding paragraph --for $a > 1$--if $r < r_1$ then $a^r < a^{r_1}$. So, a^{r_1}--that is $(\sqrt[p]{a})^{r_1 p}$--is an upper bound of $\{y: \exists_{r < r_1}\, y = a^r\}$. To prove that a^{r_1} is, in fact, the

least upper bound, we make use of the continuity of the exponential function with base a and rational arguments and prove that if $b < a^{r_1}$ then b is not an upper bound of the set in question. In fact, suppose that $b < a^{r_1}$. Then, $a^{r_1} - b$ is a positive number--say c--and, by continuity, there is a $d > 0$ such that

$$\forall_r [\,|r - r_1| < d \Rightarrow |a^r - a^{r_1}| < c].$$

Now,

$$|r - r_1| < d \Longleftrightarrow -d \leq r - r_1 < d$$

and

$$|a^r - a^{r_1}| < c \Longleftrightarrow -c < a^r - a^{r_1} < c.$$

In particular,

$$\text{if } -d < r - r_1 < 0 \text{ then } |r - r_1| < d$$

and

$$\text{if } |a^r - a^{r_1}| < c \text{ then } -c < a^r - a^{r_1}.$$

Consequently, by continuity,

$$\text{if } -d < r - r_1 < 0 \text{ then } -c < a^r - a^{r_1}$$

--that is,

$$\text{if } r_1 - d < r < r_1 \text{ then } a^r > a^{r_1} - c = b.$$

Now, for each $r < r_1$, a^r belongs to $\{y: \exists_{r < r_1} y = a^r\}$ and, so, for each r such that $r_1 - d < r < r_1$, a^r is a member of the set in question which is greater than b. If there is such a number r, it will follow that there is a member of the set which is greater than b--that is, that b is not an upper bound of the set. And, that there is, indeed, such a number r follows, since $d > 0$, from Theorem 201.

Consequently, since $(\sqrt[p]{a})^{r_1 p}$ is an upper bound of $\{y: \exists_{r < r_1} y = a^r\}$ and since no smaller number is an upper bound of this set, it follows that the least upper bound of $\{y: \exists_{r < r_1} y = a^r\}$ is $(\sqrt[p]{a})^{r_1 p}$. So, (4_1) is consistent with (3).

The situation with respect to:

$$(4_2) \qquad \forall_{0 < x < 1}\, x^u = (1/x)^{-u}$$

is, now, much simpler. The problem of meaningfulness is settled by noting that, for $0 < a < 1$, $1/a > 1$ and, so, that, for any b, $(1/a)^{-b}$ is determined by (4_1). The problem of the consistency of (4_2) also reduces to that of (4_1) via Theorem 210.

PROOF OF THEOREM 202

What we need to prove is, mainly, that, for $a > 1$, the exponential function with base a is an increasing continuous function whose range is the set of all positive numbers. The proof of this will, of course, depend on:

$$(4_1) \qquad \forall_{x > 1} \forall_u x^u = \text{the least upper bound of } \{y: \exists_{r < u} y = x^r\}$$

Since (4_1) defines the exponential functions with bases *greater* than 1 for all real numbers as arguments, the proof referred to will settle Theorem 202 for all *such* functions. Once this is done, it will follow, using:

$$(4_2) \qquad \forall_{0 < x < 1} \forall_u x^u = (1/x)^{-u}$$

and Theorem 164, that, for $0 < a < 1$, the exponential function with base a is a decreasing continuous function whose range is the set of all positive numbers. [The proof is like that in the paragraph on page 9-254 which immediately precedes '*Remark.*' The proof, on these lines, of the decreasing character of the functions in question uses Theorem 94.] Since it has already been noted that, by (4_3), the exponential function with base 1 is continuous and is defined for all real numbers, this will complete the proof of Theorem 202.

We begin by proving that, for $a > 1$, the exponential function with base a is an increasing function. [To do so, we make use of the corresponding property of the exponential function with base a and rational arguments, and of Theorem 201.] Suppose that $u_2 > u_1$. By Theorem 201, there are rational numbers r_1 and r_2 such that $u_2 > r_2 > r_1 > u$. Since the exponential function with base $a > 1$ has been shown to be increasing on the set of rational numbers, $a^{r_2} > a^{r_1}$. Now, from (4_1), a^{u_2} is an upper bound of $\{y: \exists_{r < u_2} y = x^r\}$. Consequently, since $r_2 < u_2$, $a^{u_2} \geq a^{r_2}$. Hence, $a^{u_2} > a^{r_1}$. Also, from (4_1), a^{u_1} is the *least* upper bound of $\{y: \exists_{r < u_1} y = x^r\}$. Since $r_1 > u_1$, it follows that if $r < u_1$ then $r_1 > r$. Since the exponential function with base a is increasing on the set of rational numbers, it follows that if $r < u_1$ then $a^{r_1} > a^r$--that is, that a^{r_1} is *an* upper bound of $\{y: \exists_{r < u_1} y = a^r\}$. Consequently, $a^{r_1} \geq a^{u_1}$. Since, as proved above, $a^{u_2} > a^{r_1}$, it follows that $a^{u_2} > a^{u_1}$. So, for $a > 1$,

$$\forall_{u_1} \forall_{u_2} [u_2 > u_1 \Rightarrow a^{u_2} > a^{u_1}]$$

--that is, for a > 1, the exponential function with base a is an increasing function.

Next, we shall show that, for a > 1, the exponential function with base a is continuous--that is, that, for any u_0, given c > 0, there is a d > 0 such that

$$\forall_u [|u - u_0| < d \Rightarrow |a^u - a^{u_0}| < c].$$

It would be natural to use the same procedure we used in proving that, for a > 1, the exponential function with base a, restricted to rational arguments, is continuous. In fact, the proof for continuity at 0 will go through just as before--all that is needed is to replace 'r' by 'u' and make use of the result just proved that, for a > 1, the exponential function with base a is increasing on the set of all real numbers. However, the remainder of the earlier proof makes use of Theorem 205_r, and, since we have not yet proved Theorem 205 [and intend to use Theorem 202 in proving it], this part of the proof cannot be carried over to the present situation. So, we must seek a different proof. As previously indicated [see Remark on page 9-254], we shall use the result that, for a > 1, given r_1 and c > 0 there is a d > 0 such that

$$(***) \qquad \forall_{r_0 \leq r_1} \forall_r [|r - r_0| < d \Rightarrow |a^r - a^{r_0}| < c].$$

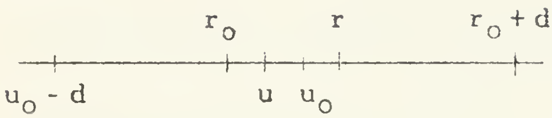
To apply this result in proving that the exponential function with base a is continuous at u_0, we choose for r_1 some rational number greater than u_0. [Given c > 0, a d > 0 which works for all rational numbers less than or equal to r_1 <u>should</u> also work for any number u_0, rational or not, which is less than r_1.]

Suppose, then, for a > 1 and some number u_0, we are given a number c > 0, have chosen an $r_1 > u_0$, and that d is positive and satisfies (***). We proceed to show that, for any u,

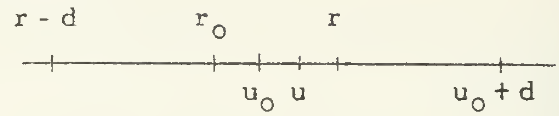
$$\text{if } |u - u_0| < d \text{ then } |a^u - a^{u_0}| < c.$$

To do so, suppose that $|u - u_0| < d$. Ignoring the trivial case in which $u = u_0$--in which case, $|a^u - a^{u_0}| = 0 < c$--there are two cases to consider:

<u>Case 1.</u> $u_0 - d < u < u_0$



<u>Case 2.</u> $u_0 < u < u_0 + d$



In Case 1, let r_0 be a rational number such that $u_0 - d < r_0 < u$. Since $r_0 < u < u_0 < r_1$, $r_0 \leq r_1$ and (***) applies. Since $u_0 - d < r_0$, $u_0 < r_0 + d$. Let r be some rational number such that $u_0 < r < r_0 + d$. Then since

$$(1) \qquad r_0 < u < u_0 < r < r_0 + d,$$

it follows that
$$0 < r - r_0 < d$$
and, in particular, that $|r - r_0| < d$. Consequently, by (***), $|a^r - a^{r_0}| < c$. Now, since the exponential function with base a is increasing, it follows from (1) that

$$a^{r_0} < a^u < a^{u_0} < a^r.$$

Consequently,

$$0 < a^{u_0} - a^u < a^r - a^{r_0}$$

and, so,

$$|a^u - a^{u_0}| < |a^r - a^{r_0}|.$$

Since $|a^r - a^{r_0}| < c$, it follows [as was to be proved] that $|a^u - a^{u_0}| < c$.

The second case is treated similarly. This time, let r be a rational number such that $u < r < u_0 + d$ and let r_0 be one such that $r - d < r_0 < u_0$. Since $r_0 < u_0 < r_1$, $r_0 \leq r_1$ and (***) applies. Since

$$(2) \qquad r - d < r_0 < u_0 < u < r,$$

it follows that
$$-d < r_0 - r < 0$$
and, in particular, that $|r - r_0| < d$. Consequently, by (***), $|a^r - a^{r_0}| < c$. Again, since the exponential function with base a is increasing, it follows from (2) that

$$a^{r_0} < a^{u_0} < a^u < a^r.$$

Consequently,

$$0 < a^u - a^{u_0} < a^r - a^{r_0}$$

and, so,

$$|a^u - a^{u_0}| < |a^r - a^{r_0}|.$$

Since $|a^r - a^{r_0}| < c$, it follows [as was to be proved] that $|a^u - a^{u_0}| < c$.

Consequently, in any case, for $a > 1$ and any u_0, given $c > 0$, there is a $d > 0$ such that

$$\text{if } |u - u_0| < d \text{ then } |a^u - a^{u_0}| < c$$

--for $a > 1$, the exponential function with base a is continuous.

The final step needed to complete the proof of Theorem 202 is the proof that, for $a > 1$, the range of the exponential function with base a is the set of all positive numbers. There are two things to be established,

> first, that each value of the function in question is positive, and
> second, that each positive number is a value of the function.

To prove the first, we use (4_1) and Theorem 203_r. For any u, let r be some rational number less than u. By (4_1), $a^r \leq a^u$. By Theorem 203_r, $a^r > 0$. Consequently, $a^u > 0$.

To prove the second, let y be any positive number, and choose a number M which is greater than both y and $1/y$. Then, $M > 0$ and, for a given $a > 1$, there is an N such that, for each $r > N$,

$$a^r > M \text{ and } a^{-r} < 1/M.$$

Choose some such rational number r. Since $M > y$, $a^r > y$ and since $1/M < y$, $a^{-r} < y$. Now, consider the function

$$\{(u,\ y),\ |u| \leq r : y = a^u\}$$

whose domain is the segment $\overline{-r,\ r}$. Since, for $a > 1$, the exponential function with base a is an increasing continuous function, the same is true of its restriction to $\overline{-r,\ r}$. So, by Theorem 187, the range of this latter function is $\overline{a^{-r},\ a^r}$. Since, by the choice of r, y belongs to this range, it also belongs to the range of the whole exponential function.

This completes the proof of Theorem 202.

PROOFS OF THEOREMS 203-211

The first and last of these theorems--Theorems 203 and 211--are almost immediate corollaries of Theorem 202. For, by Theorem 202, for $0 < a \neq 1$, the range of the exponential function with base a is $\{y: y > 0\}$. So, for each <u>such</u> base, $\forall_u\ a^u > 0$. And, by (4_3), $\forall_u\ 1^u = 1 > 0$. So, Theorem 203. As to Theorem 211, this merely says that, for $0 < a \neq 1$, the exponential function with base a has an inverse. Since,

by Theorem 202, each such function is monotonic, Theorem 211 follows by Theorem 184.

Of the remaining theorems, Theorems 206, 209, and 210 follow, as in the rational case, from preceding theorems. So, all we need prove here are Theorems 204, 205, 207, and 208.

Each of these theorems can be derived from the corresponding theorem for rational exponents by using the fact that the exponential functions are continuous [Theorem 202], together with some standard theorems about continuous functions. The chief of these latter theorems is the following:

> If f and g are continuous functions whose domain is the set of all real numbers and are such that, for each rational number r, $f(r) = g(r)$, then $f = g$ --that is, $\forall_u\, f(u) = g(u)$.

In brief, there is <u>at most</u> one continuous function which has prescribed values for all rational numbers and which is defined for all real numbers. [Whether or not there <u>is</u> such a function depends on how its values are prescribed at the rational points of the number line. Since each subset of a continuous function is a continuous function, a necessary condition that it be possible to extend a function whose domain is the rationals to a continuous function whose domain is the real numbers is that the original function be continuous. It turns out that a necessary and sufficient condition is that the original function be uniformly continuous on each interval of rational numbers.]

The proof of this uniqueness theorem is quite easy. Suppose that f and g are continuous functions defined for all real numbers and such that, for each r, $f(r) = g(r)$. Given any real number u_o and <u>any</u> $c > 0$, there is, since f and g are continuous at u_o, a $d > 0$ such that

$$\forall_u [\, |u - u_o| < d \Rightarrow (|f(u) - f(u_o)| < \tfrac{c}{2} \text{ and } |g(u) - g(u_o)| < \tfrac{c}{2})].$$

Let r be some rational number such that $|r - u_o| < d$ [Theorem 201]. Then,

$$|f(r) - f(u_o)| < \tfrac{c}{2} \text{ and } |g(r) - g(u_o)| < \tfrac{c}{2}.$$

But, by hypothesis, $f(r) = g(r)$. So, $f(u_o)$ and $g(u_o)$ each differs by less than $c/2$ from the common value of f and g at r. Consequently, $f(u_o)$

and $g(u_o)$ differ by less than c--that is,

$$|f(u_o) - g(u_o)| < c.$$

[More formally, since $|f(u_o) - f(r)| = |f(r) - f(u_o)|$, it follows, by Theorem 169c, that $|[f(u_o) - f(r)] + [g(r) - g(u_o)]| \leq |f(r) - f(u_o)| + |g(r) - g(u_o)| < \frac{c}{2} + \frac{c}{2} = c$. So, since $f(r) = g(r)$, $|f(u_o) - g(u_o)| < c$.] It follows from this that $|f(u_o) - g(u_o)| \not< c$ and, so, we have proved that $|f(u_o) - g(u_o)|$ is not any positive number. Since $|f(u_o) - g(u_o)|$ cannot be negative, it follows that $|f(u_o) - g(u_o)| = 0$. Consequently, $f(u_o) = g(u_o)$.

The method used in proving Theorems 204, 205, 207, and 208 shows up most clearly in the proof of the last of these theorems--so, we shall begin by proving Theorem 208. In the proof we shall need to use a standard theorem on continuous functions:

<u>The product of two continuous functions is continuous.</u>

We shall defer the proof of this, and other such theorems which we shall use, until the completion of the proofs of Theorems 203-211. Aside from this theorem, we shall use only Theorem 208_r, Theorem 202, and the uniqueness theorem which is proved in the preceding paragraph.

To prove Theorem 208 we need to show that, for any $a > 0$ and any $b > 0$,

$$\forall_u \; (ab)^u = a^u b^u.$$

To do this, consider, for any given $a > 0$ and $b > 0$, the functions f and g defined, for each real number u, by:

$$f(u) = (ab)^u \qquad \text{and:} \qquad g(u) = a^u b^u$$

We wish to prove that, for each u, $f(u) = g(u)$--that is, that $f = g$. Now, by Theorem 208_r, we know that, for each rational number r, $f(r) = g(r)$. So, we can conclude, by the uniqueness theorem, that $f = g$ once we have shown that both f and g are continuous. Now, f is merely the exponential function with base ab [since $a > 0$ and $b > 0$, $ab > 0$]. So, by Theorem 202, f is continuous. And g is also continuous since it is the product of two functions each of which is, again by Theorem 202, continuous. Consequently, Theorem 208.

The proof of Theorem 204 is just about as simple. Again, we need a standard theorem about continuous functions, whose proof we defer:

<u>The reciprocal of a continuous function is continuous.</u>

[Recall that an argument at which a function is 0 does not belong to the domain of its reciprocal.] Aside from this theorem, we shall use only Theorem 204_r, Theorem 202, and the uniqueness theorem.

To prove Theorem 204 we need to show that, for any $a > 0$,

$$\forall_u\, a^{-u} = \frac{1}{a^u}.$$

To do this, consider, for any given $a > 0$, the functions f and g which are defined, for each real number u, by:

$$f(u) = a^{-u} \quad \text{and:} \quad g(u) = \frac{1}{a^u}$$

Note that, by Theorem 202, $a^u \neq 0$, for each u.]

We wish to prove that, for each u, $f(u) = g(u)$--that is, that $f = g$. Now, by Theorem 204_r, we know that, for each rational number r, $f(r) = g(r)$. So, we can conclude, by the uniqueness theorem, that $f = g$ once we have shown that both f and g are continuous.

Consider the function f. Since $a > 0$, it follows from Theorem 202 that, given any real number u_o, the exponential function with base a is continuous at $-u_o$. So, given $c > 0$, there is a $d > 0$ such that

$$\text{if } \quad |-u - -u_o| < d \quad \text{then} \quad |a^{-u} - a^{-u_o}| < c.$$

But, $|-u - -u_o| = |u - u_o|$. So, by the definition of f, this is equivalent to:

$$\text{if } \quad |u - u_o| < d \quad \text{then} \quad |f(u) - f(u_o)| < c$$

So, f is continuous at each of its arguments u_o.

Consider the function g. This is just the reciprocal of the exponential function with base a which, by Theorem 202, is a continuous function. So, by the standard theorem quoted above, g is continuous.

Consequently, Theorem 204.

The proof of Theorem 205 is similar to those of Theorem 204 and 208, but consists of two parts. The first part consists in using Theorem 205_r:

$$\forall_{x>0} \forall_r \forall_s\, x^r x^s = x^{r+s}$$

[and Theorem 202 and the uniqueness theorem] to prove:

$$(*) \qquad \forall_{x>0} \forall_r \forall_v\, x^r x^v = x^{r+v}$$

The second part consists in using (*) to prove Theorem 205:

$$\forall_{x>0}\forall_u\forall_v\, x^u x^v = x^{u+v}$$

To prove (*), we must show that if, for any $a > 0$ and any r,

$$f(v) = a^r a^v \qquad \text{and} \qquad g(v) = a^{r+v}$$

then $f = g$--that is, for each v, $f(v) = g(v)$. As usual, a previous theorem--in this case, Theorem 205_r--tells us that, for each s, $f(s) = g(s)$. To complete the proof of (*) all we need show is that f and g are continuous.

Consider f. This function is the product of the constant function whose only value is a^r and the exponential function with base a. Each constant function is continuous and, by Theorem 202, since $a > 0$, the exponential function with base a is continuous. So, by our standard theorem on the product of continuous functions, f is continuous. [Actually, the proof that the product of a constant function by a continuous function is continuous is extremely easy.]

Consider g. Since $a > 0$, it follows that, given any real number v_0, the exponential function with base a is continuous at $r + v_0$. So, given $c > 0$, there is a $d > 0$ such that

$$\text{if } |(r+v) - (r+v_0)| < d \text{ then } |a^{r+v} - a^{r+v_0}| < c.$$

But, $|(r+v) - (r+v_0)| = |v - v_0|$. So, by the definition of g, this is equivalent to:

$$\text{if } \qquad |v - v_0| < d \text{ then } |g(v) - g(v_0)| < c$$

So, g is continuous at each of its arguments v_0.

Consequently (*).

Now, to prove Theorem 205, we proceed in just the same way, using (*) instead of Theorem 205_r. For $a > 0$ and any given v, let

$$f(u) = a^u a^v \qquad \text{and} \quad g(u) = a^{u+v}$$

By (*), $f(r) = g(r)$, for each r, and, just as in the first part of the proof, it is easy to see that f and g are continuous. So, by the uniqueness theorem, for each u, $f(u) = g(u)$.

Consequently, Theorem 205.

Finally, we need to prove Theorem 207. Once again we need a standard theorem on continuous functions--this time:

<u>The composition of two continuous functions is continuous.</u>

As in the case of Theorem 205, the proof is in two parts. In the first part, we use Theorem 207_r:

$$\forall_{x>0}\forall_r\forall_s (x^r)^s = x^{rs}$$

to prove:

$$(*) \qquad \forall_{x>0}\forall_u\forall_s (x^u)^s = x^{us}$$

In the second part, we use (*) to prove Theorem 207:

$$\forall_{x>0}\forall_u\forall_v (x^u)^v = x^{uv}$$

To prove (*), consider any $a > 0$ and any rational number s, and let

$$f(u) = (a^u)^s \text{ and } g(u) = a^{us}.$$

By Theorem 207_r, for each r, $f(r) = g(r)$. So, as usual, we can show that, for each u, $f(u) = g(u)$--that is, we can prove (*)--merely by showing that f and g are continuous.

Consider f. Since $s \in R$, there is a $p \in I^+$ such that $sp \in I$ and, by definition, $f(u) = \left(\sqrt[p]{a^u}\right)^{sp}$. So, f is the composite of three functions--the exponential function with base a [$a > 0$], the pth principal root function, and the (sp)th power function. The first of these is continuous by Theorem 202, the second by Theorem 189. As to the third, it is continuous by Theorem 186 if $sp > 0$, is constant [and, so, continuous] if $sp = 0$, and is the reciprocal of a positive-integral power function [and, so, by Theorem 186 and a standard theorem, continuous] if $sp < 0$. Consequently, f is continuous.

Consider g. If $s = 0$, g is constant and, so, continuous. Suppose that $s \neq 0$. Since $a > 0$, it follows that, given any real number v_o, the exponential function with base a is continuous at $v_o s$. So, given a $c > 0$, there is a $d > 0$ such that

$$\text{if } |vs - v_o s| < d \text{ then } |a^{vs} - a^{v_o s}| < c.$$

But, $|vs - v_o s| = |v - v_o| \cdot |s|$ and $|s| \neq 0$. So, by the definition of g, this is equivalent to:

$$\text{if } |v - v_o| < d/|s| \text{ then } |g(v) - g(v_o)| < c.$$

So [since, for $d > 0$, $d/|s| > 0$], g is continuous at each of its arguments v_o.

Consequently, by the uniqueness theorem (*).

Now, to prove Theorem 207, we proceed in the usual fashion, using (*). For $a > 0$, and any given u, let

$$f(v) = (a^u)^v \text{ and } g(v) = a^{uv}.$$

By (*), $f(s) = g(s)$ for each s. f is the exponential function with base a^u and, so, by Theorem 202, is continuous. The proof that g is continuous is just like the corresponding portion of the first part of the proof. So by the uniqueness theorem, for each u, $f(u) = g(u)$. Consequently, Theorem 207.

This completes the proof of Theorems 203-211.

THREE STANDARD THEOREMS ON CONTINUITY

For completeness, we prove the three standard theorems we have used on continuous functions:

(1) The product of two continuous functions is continuous.

(2) The reciprocal of a continuous function is continuous.

(3) The composition of two continuous functions is continuous.

<u>Proof of (1)</u>. Since

$$f(u)g(u) - f(u_o)g(u_o) = [f(u) - f(u_o)]g(u) + f(u_o)[g(u) - g(u_o)],$$

it follows that

$$|f(u)g(u) - f(u_o)g(u_o)| \le |f(u) - f(u_o)| \cdot |g(u)| + |f(u_o)| \cdot |g(u) - g(u_o)|.$$

Now, suppose that g is continuous at u_o. Then, there is a $d_1 > 0$ such that, if $|u - u_o| < d_1$ then $|g(u) - g(u_o)| < 1$. Since $|g(u)| - |g(u_o)| \le |g(u) - g(u_o)|$, it follows that, if $|u - u_o| < d_1$, then $|g(u)| < 1 + |g(u_o)|$. Let M be the larger of $1 + |g(u_o)|$ and $|f(u_o)|$. Then, for $|u - u_o| < d_1$,

$$|f(u)g(u) - f(u_o)g(u_o)| \le |f(u) - f(u_o)| \cdot M + |g(u) - g(u_o)| \cdot M.$$

Now, suppose that both f and g are continuous at u_o. It follows that, given $c > 0$, there are numbers $d' > 0$ and $d'' > 0$ such that

$$\text{if } |u - u_o| < d' \text{ then } |f(u) - f(u_o)| < c/(2M)$$

and

$$\text{if } |u - u_0| < d'' \qquad \text{then} \quad |g(u) - g(u_0)| < c/(2M).$$

Consequently, if d is the smallest of d_1, d' and d'' then $d > 0$ and

$$\text{if } |u - u_0| < d \text{ then } |f(u)g(u) - f(u_0)g(u_0)| < \frac{c}{2M} \cdot M + \frac{c}{2M} \cdot M = c$$

--that is, the product of f and g is continuous at u_0.

Proof of (2). Suppose that u_0 is an argument of the reciprocal of g. Then $g(u_0) \neq 0$ and, hence, $|g(u_0)|/2 > 0$. So, assuming that g is continuous at u_0, there is a $d_1 > 0$ such that, for $|u - u_0| < d_1$, $|g(u) - g(u_0)| < |g(u_0)|/2$. Since $|g(u_0)| - |g(u)| \leq |g(u_0) - g(u)| = |g(u) - g(u_0)|$, it follows that, for $|u - u_0| < d_1$, $|g(u)| > |g(u_0)|/2$--and, in particular, $g(u) \neq 0$.

Now,

$$\left| \frac{1}{g(u)} - \frac{1}{g(u_0)} \right| = \frac{|g(u) - g(u_0)|}{|g(u)| \cdot |g(u_0)|}$$

and, for $|u - u_0| < d_1$, it follows that

$$\left| \frac{1}{g(u)} - \frac{1}{g(u_0)} \right| < \frac{|g(u) - g(u_0)|}{[|g(u_0)|/2]^2}$$

Now, since g is continuous at u_0, given $c > 0$, there is a $d' > 0$ such that

$$\text{if } |u - u_0| < d' \text{ then } |g(u) - g(u_0)| < c[|g(u_0)|/2]^2.$$

Consequently, if d is the smaller of d_1 and d' then $d > 0$ and

$$\text{if } |u - u_0| < d \text{ then } \left| \frac{1}{g(u)} - \frac{1}{g(u_0)} \right| < c$$

--that is, the reciprocal of g is continuous at u_0.

Proof of (3). Suppose that $u_0 \in \mathcal{D}_g$, $g(u_0) \in \mathcal{D}_f$, g is continuous at u_0, and f is continuous at $g(u_0)$. Then, given $c > 0$, there is a $d' > 0$ such that if $|v - g(u_0)| < d'$ then $|f(v) - f(g(u_0))| < c$ and since $d' > 0$, there is a $d > 0$ such that if $|u - u_0| < d$ then $|g(u) - g(u_0)| < d'$. Consequently,

$$\text{if } |u - u_0| < d \text{ then } |f(g(u)) - f(g(u_0))| < c$$

--that is, the composition of f with g is continuous at u_0.

CONTINUITY OF POWER FUNCTIONS

As another application of (3) we shall use it and some theorems on exponential and logarithm functions to prove:

> Theorem 220.
>
> For each u, the power function with exponent u and positive arguments is continuous.

By Theorem 217, for any $a > 0$,

$$\log (a^u) = u \log a.$$

So, by the defining principle (L),

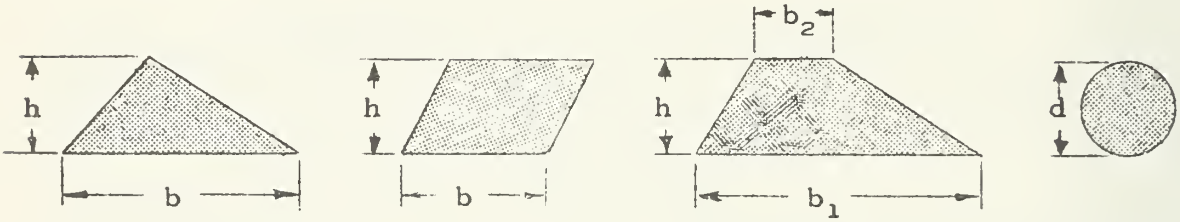
$$\left.\begin{aligned} a^u &= 10^{u \log a} \\ &= (10^u)^{\log a} \end{aligned}\right\} \text{Theorem 207}$$

Consequently, if exp is the exponential function with base 10^u, it follows that the power function with exponent u and positive arguments is $\exp \circ \log$. Since both exp and log are continuous [Theorems 202 and 213], so, by (3), is the power function.

APPENDIX D

[This appendix contains an abbreviated development of the usual volume and surface area formulas for simple solids.]

Volume-measures.--In Unit 6 you derived formulas for the area-measures of plane regions of certain kinds. For example:



$K = \frac{1}{2}bh$ $\quad K = bh$ $\quad K = \frac{1}{2}(b_1 + b_2)h$ $\quad K = \frac{\pi d^2}{4}$

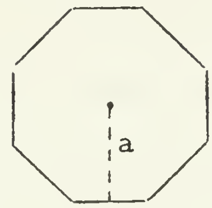
[The formulas in the first three examples which deal with polygonal regions are derived by using two area-measure axioms, Axioms I and J.

> Axiom I.
>
> The area-measure of a triangular region is half the product of the measure of any of its sides by the measure of the altitude from the opposite vertex of the triangle.

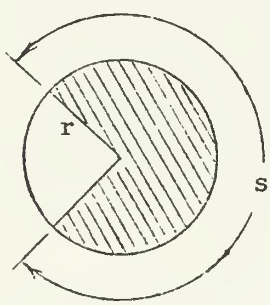
> Axiom J.
>
> The area-measure of a polygonal region is the sum of the area-measures of any set of component triangular regions into which it can be cut.

The fourth formula, for the area-measure of a circular region, was not proved, but was made to appear reasonable by considering the area-measures of inscribed polygonal regions. In brief, using Axioms I and J, it was proved that the area-measure of a region bounded by a regular

polygon is $\frac{1}{2}ap$, where a is the apothem [the radius of the inscribed



circle] of the polygon and p is its perimeter. Now, suppose that c is a circle of diameter d and, for each n, P_n is a regular n-gon, inscribed in c, whose apothem is a_n and whose perimeter is p_n. It seemed reasonable [in Unit 6] that, for sufficiently large values of n, a_n differs as little as one may wish from the radius, d/2, of c, and that p_n differs as little as one may wish from the circumference, πd, of c. Consequently, for n sufficiently large, the area-measure, $\frac{1}{2}a_n p_n$, of the region bounded by P_n differs as little as one may wish from $\frac{1}{2} \cdot \frac{d}{2} \cdot \pi d$--that is, from $\frac{\pi d^2}{4}$. Since any given point belonging to the circular region is, if n is sufficiently large, inside P_n, it is natural to define the area-measure of the circular region to be $\frac{\pi d^2}{4}$. Similar considerations suggested adopting the formula:



$$K = \frac{1}{2}rs$$

for the area-measure of a circular sector of radius r and arc-measure s.]

In this Appendix we shall give informal justifications for some formulas for the volume-measures of some simple "solid" regions, including prisms, cylinders, pyramids, cones, and solid spheres, and for some formulas for the area-measures of the surfaces of such solids. We shall not attempt to base these justifications entirely on formally-stated axioms. Instead, we shall assume that you can recognize when two solids are congruent--that is, when they have the same size and shape--and that you agree that, given a solid, there is a solid congruent to it anywhere you please--in other words, that a solid can be "moved" anywhere you like without changing its size and shape. We shall also assume that congruent solids have the same volume-measure, and that if a

solid is cut up into other solids, the volume-measure of the given solid is the sum of the volume-measures of its parts. Finally, we shall state, later, two numerical axioms on volume-measures.

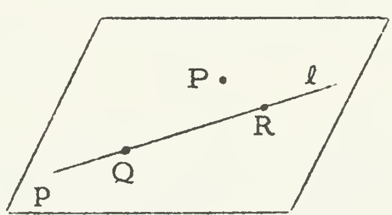
Before we go into the matter of formulas for volumes and surface areas of solids, let's spend a little time in becoming acquainted with a few simple concepts about points, lines, and planes. These ideas will be useful to us in our discussions of the mensuration formulas. [As you read and work exercises, practice drawing three-dimensional illustrations.]

POINTS, LINES, AND PLANES

Consider three noncollinear points A, B, and C. These points are the vertices of a triangle, $\triangle ABC$. Pick any two points of $\triangle ABC$. These points determine a line. Now, consider the union of all such lines. This union is called <u>the plane ABC</u>, or the plane <u>determined</u> by the three noncollinear points A, B, and C.

[We shall use capital letters as variables whose domain is the set of points of space, the letters 'ℓ', 'm', and 'n' as variables whose domain is the set of all lines, and the letters 'p', 'q', and 'r' as variables whose domain is the set of all planes.]

Given a line ℓ, there are many planes which contain ℓ. But, given a line ℓ and a point P such that $P \notin \ell$, there is exactly one plane p such that $\ell \subseteq p$ and $P \in p$. If Q and R are two points on ℓ then p = plane PQR.

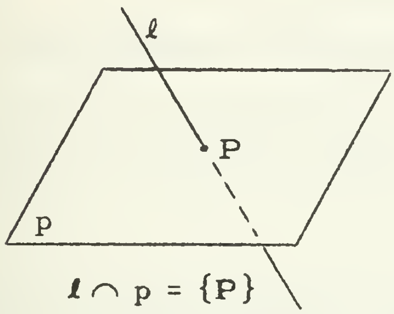


(1) Given a circle c with diameter $\overline{AB}$. How many planes contain c? How many planes contain $\overline{AB}$?

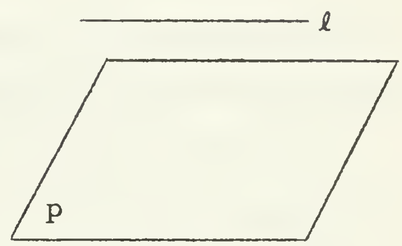
(2) Suppose that $A \in p$ and $B \in p$. Does it follow that $\overline{AB} \subseteq p$? That $\overline{AB} \subseteq p$? That $\overrightarrow{AB} \subseteq p$? That $\overleftrightarrow{AB} \subseteq p$?

(3) Suppose that a line ℓ is not contained in a plane p. How many points are there in $\ell \cap p$?

(4) A line ℓ is said to <u>intersect</u> a plane P if and only if $\ell \cap p$ consists of exactly one point. A line ℓ is said to be <u>parallel</u> to a plane p $[\ell \parallel p]$ if and only if $\ell \cap p = \emptyset$. Suppose that $\ell \cap p$ consists of more than one point, what can you say about ℓ and p?

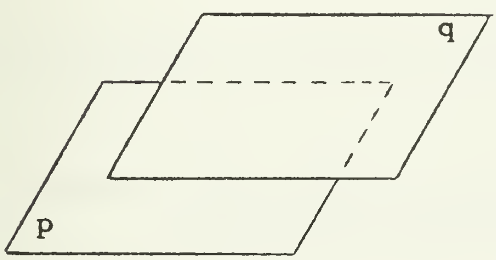


$\ell \cap p = \{P\}$

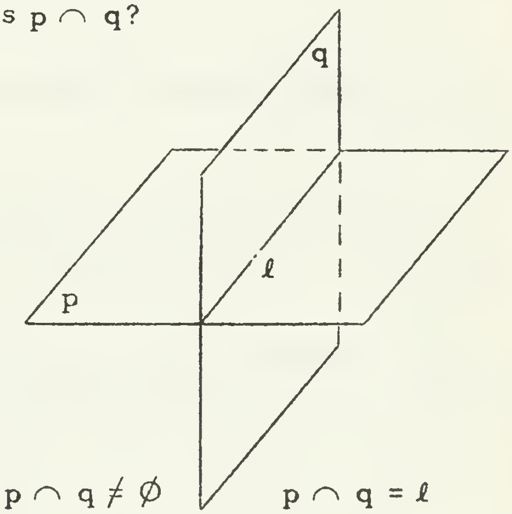


$\ell \cap p = \emptyset$

(5) Planes p and q are said to be <u>parallel</u> if and only if $p \cap q = \emptyset$. If $p \nparallel q$, what kind of geometric figure is $p \cap q$?



$p \cap q = \emptyset$



$p \cap q \neq \emptyset$ $\quad$ $p \cap q = \ell$

(6) Given a plane p, how many planes q are there such that $q \parallel p$? Given a plane p and a line ℓ such that $\ell \parallel p$. How many planes q are there such that $\ell \subseteq q$ and $q \parallel p$?

(7) Given a plane p and a line ℓ such that $\ell \nparallel p$. How many planes q are there such that $\ell \subseteq q$ and $q \parallel p$? How many planes q are there such that $\ell \subseteq q$ and $q \nparallel p$?

(8) Suppose that $\ell \parallel p$, $\ell \subseteq q$, and $p \cap q \neq \emptyset$. Draw a sketch illustrating this supposition. Suppose, further, that $\ell' = p \cap q$. Are ℓ and ℓ' subsets of the same plane? Are they parallel? [Parallel lines are nonintersecting lines which are contained in one plane.]

(9) Suppose that $\ell \cap p = \{P\}$ and $\ell \subseteq q$. Draw a sketch illustrating this supposition. What can you say about $p \cap q$? What can you say about $\ell \cap (p \cap q)$?

(10) Given a line ℓ and a point $P \in \ell$. Let p be a plane which contains ℓ, and m be the line in p such that m is perpendicular to ℓ at P. Let q be another plane which contains ℓ, and n be the line in q such that n is perpendicular to ℓ at P. Does $m = n$?

(11) How many lines are there which are perpendicular to a given line at a given point on the line?

(12) How many lines are there which are perpendicular to a given line and which contain a given point not on the given line?

(13) Given a line ℓ and a point $P \in \ell$. Consider the set of all lines m such that $m \perp \ell$ at P. What kind of figure is the union of all such lines m?

(14) Given a line ℓ. Consider the set of all lines m such that $m \perp \ell$. What is the union of all such lines m?

(15) Suppose that $\ell \cap p = \{F\}$ and, for each line $m \subseteq p$ for which $F \in m$, $m \perp \ell$. Draw a sketch illustrating this supposition.

*

For each line ℓ and for each plane p such that ℓ intersects p, ℓ is said to be perpendicular to p if and only if ℓ is perpendicular to each line of p which contains the point of intersection of ℓ and p. The point of intersection is said to be the foot of the perpendicular.

*

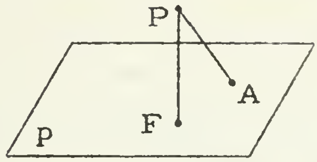
(16) a. Given a plane p and a point $P \notin p$. How many lines ℓ are there such that $P \in \ell$ and

1. ℓ intersects p? 2. $\ell \perp p$? 3. $\ell \parallel p$?

b. Given a plane p and a point $P \in p$. How many lines ℓ are there such that $P \in \ell$ and

1. ℓ intersects P? 2. $\ell \perp P$? 3. $\ell \parallel P$?

(17) Suppose that $P \notin p$, $F \in p$, and $\overleftrightarrow{PF} \perp p$. If A is any point of p other than F, which of the segments $\overline{PF}$ and $\overline{PA}$ is the longer? If PF = 8 and FA = 6, compute PA.



*

The measure of the segment joining a point P and the foot of the perpendicular through P to a plane p is the <u>distance</u> between P and p. [What is meant by 'the distance between a point P and a line ℓ'?]

*

(18) Suppose that $\triangle ABC$ is contained in a plane p. $\angle C$ is a right angle, AC = BC, and $AB = 5\sqrt{2}$. If $P \notin p$ and $\overleftrightarrow{PC} \perp p$, show that PA = PB. Also, if PA = 13, what is the distance between P and p, and what is the distance between P and $\overleftrightarrow{AB}$?

(19) What is meant by 'the distance between two parallel lines'? What do you think is meant by 'the distance between two parallel planes'?

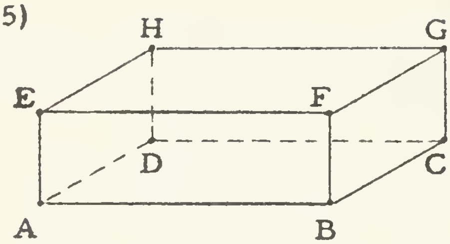
(20) Suppose that the area-measure of the square region ABCD is s^2. Let p and q be parallel planes such that $\overline{AB} \subseteq p$ and $\overline{CD} \subseteq q$. What can you say about the distance between p and q?

(21) Given a point A at a distance d from a plane p. Let F be the foot of the perpendicular from A to p. Let B and C be points in p such that $\overline{BF} \perp \overline{CF}$, $\angle ACF$ is a 60°-angle, and, BF = d. Compute AB, AC, CF, and BC.

(22) Suppose that F is the foot of the perpendicular from a point P to a plane p. Let $\triangle ABC$ be an equilateral triangle contained in p. If F is the centroid of $\triangle ABC$, and AB = 10 = AP, find the distance between P and p.

(23) The foot of the perpendicular from a point P to a plane p is at distances a and b from points A and B of p, respectively. How far is P from p if $PA^2 + PB^2 = (a + b)^2$?

(24) Given parallel planes p and q and two points B and C on p. Where must a point $A \in q$ be in order that the area-measure of $\triangle ABC$ be as small as possible? If K is this minimum area-measure, what is the distance between p and q?

(25) 

The diagram at the left is a picture of a brick with its eight corners labeled. Each two corners of the brick determines a line. Complete the first two rows of each table.

Pairs of lines	$\overleftrightarrow{AB}, \overleftrightarrow{EF}$	$\overleftrightarrow{AB}, \overleftrightarrow{EA}$	$\overleftrightarrow{AB}, \overleftrightarrow{CD}$	$\overleftrightarrow{AB}, \overleftrightarrow{GH}$	$\overleftrightarrow{AB}, \overleftrightarrow{DA}$	$\overleftrightarrow{AB}, \overleftrightarrow{HE}$	$\overleftrightarrow{FE}, \overleftrightarrow{GC}$
Intersection empty?	yes						
Parallel lines?	yes						

Pairs of lines	$\overleftrightarrow{EB}, \overleftrightarrow{FA}$	$\overleftrightarrow{EB}, \overleftrightarrow{HC}$	$\overleftrightarrow{EB}, \overleftrightarrow{GD}$	$\overleftrightarrow{EB}, \overleftrightarrow{GB}$	$\overleftrightarrow{EB}, \overleftrightarrow{FC}$	$\overleftrightarrow{EA}, \overleftrightarrow{GC}$	$\overleftrightarrow{EA}, \overleftrightarrow{GB}$
Intersection empty?							
Parallel lines?							

(26) Suppose that a and b are sets of points such that, for some line ℓ, a $\subseteq \ell$ and b $\subseteq \ell$. Then, by definition, a and b are collinear. Suppose that c and d are sets of points such that, for some plane p, c $\subseteq$ p and d $\subseteq$ p. Then, by definition, c and d are coplanar. Now, return to the tables, above, and in the next-to-the-last row, for each pair of lines, answer the question:

Are the lines coplanar?

(27) In Unit 6 we defined 'parallel lines' as lines whose intersection is empty. Such a definition is adequate as long as we restrict the domain of the relation of parallelism of lines to lines in some one plane. But, if we wish to extend the domain of this relation to include lines in

space, we need a new definition:

$\ell \parallel m$ if and only if $\ell \cap m = \emptyset$ and ℓ and m are coplanar

Pairs of lines which are not coplanar are called <u>skew lines</u>. Complete the tables by answering the question:

Are the lines skew?

(28) Suppose that $\ell \nparallel m$. Does it follow that ℓ and m are skew?

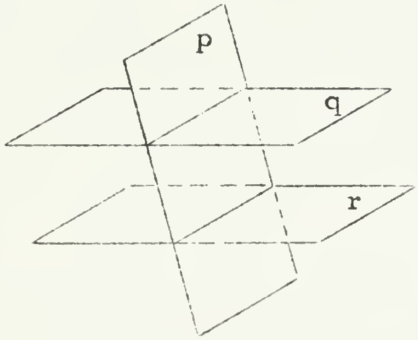
(29) Suppose that ℓ, m, and n are three lines such that $\ell \perp n$ and $m \perp n$. Under that condition would if follow that $\ell \parallel m$? That ℓ and m are skew? That ℓ and m are neither skew nor parallel?

(30) Suppose that ℓ, m, and n are three lines such that $\ell \parallel n$ and $m \parallel n$. Does it follow that $\ell \parallel m$?

(31) Suppose that p, q, and r are three planes such that $p \parallel r$ and $q \parallel r$. Does it follow that $p \parallel q$?

(32) Given that p, q, and r are planes such that $p \parallel r$ and $q \parallel r$. Under what condition would it follow that $p \nparallel q$?

(33)



Suppose that $q \parallel r$.

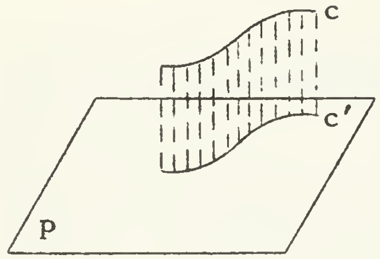
(a) If $p \cap q \neq \emptyset$, under what condition would it follow that $p \parallel r$?

(b) If $p \nparallel q$ and $p \neq q$, does it follow that $p \cap r \neq \emptyset$?

(c) If p intersects q in line ℓ and p intersects r in line m, what can you say about ℓ and m?

*

Suppose that P is a point and p is a plane, and let ℓ be the line through P perpendicular to p. The foot of the perpendicular is called <u>the projection of P on p</u>.



Suppose that c is a set of points and c' is the set of points which are the projections of the points of c on p. The set c' is called <u>the projection of c on p</u>.

*

(34) Given a line $\overleftrightarrow{AB}$ and a plane p. Suppose that $\overline{CD}$ is the projection of $\overline{AB}$ on p. Under what condition is $CD = AB$? Under what condition is $CD = 0$? $CD < AB$? $CD > AB$?

(35) Given circle c and plane p. Suppose that c' is the projection of c on p. Under what condition is $c' \cong c$? $c' = c$? c' a segment?

(36) Suppose that ℓ and m are two lines, p is a plane, and ℓ' and m' are the projections of ℓ and m on p.

(a) If $\ell \parallel p$ and $m \parallel p$, does it follow that $\ell' \cap m' = \emptyset$?

(b) If ℓ and m are skew and $m \parallel p$, does it follow that $\ell' \cap m' \neq \emptyset$?

(37) Suppose that ℓ and m are skew lines.

(a) Is there a plane through ℓ parallel to m?

(b) If $\ell \subseteq p$ and $m \parallel p$ and m' is the projection of m on p, do ℓ and m' intersect?

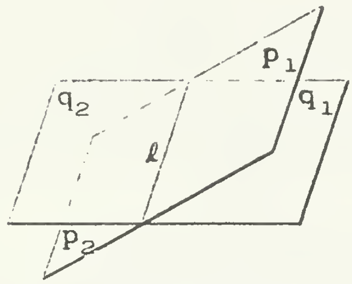
(c) Is there a line which intersects both ℓ and m and is perpendicular to each of them?

(d) How many common perpendiculars do two skew lines have?

(38) Given an isosceles right angle $\triangle ABC$, with $\angle C$ the right angle and $\overline{AC}$ 26 inches long. Suppose that p is a plane such that $C \in p$ and A and B are each 10 inches from p. If A' and B' are the projections of A and B on p, find the measure of $\angle A'CB'$.

*

Recall from Unit 6 that a line separates a plane into two half-planes. Suppose that two planes p and q intersect in the line ℓ. Let ℓ separate p into the half-planes p_1 and p_2 and ℓ separate q into the half-planes q_1 and q_2. Then, $p_1 \cap p_2 = \emptyset$ and $p_1 \cup \ell \cup p_2 = p$, and $q_1 \cap q_2 = \emptyset$ and



$q_1 \cup \ell \cup q_2 = q$. The union of two noncoplanar half-planes and their common edge is called a dihedral angle. So, for example, $p_1 \cup \ell \cup q_1$ and $q_1 \cup \ell \cup p_2$ are dihedral angles.

Consider the dihedral angle $p_1 \cup \ell \cup q_1$. Let A be a point on ℓ and let $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ be half-lines contained in p_1 and q_1, respectively, such that $\overleftrightarrow{AP} \perp \ell$ and $\overleftrightarrow{AQ} \perp \ell$. Notice that A, P, and Q are noncollinear. The angle, $\angle PAQ$, is called a plane angle of the dihedral angle and its measure is said to be the measure of the dihedral angle. A dihedral angle is called a right dihedral angle if and only if its measure is 90. What do you think an acute dihedral angle is? An obtuse dihedral angle?

The union of two intersecting planes contains four dihedral angles. Two intersecting planes are said to be perpendicular if and only if their union contains a right dihedral angle.

*

(39) Consider the dihedral angle $p_1 \cup \ell \cup q_1$. Let $\angle PAQ$ be a plane angle of this dihedral angle. If $PA = AQ = QP$, what is the measure of the dihedral angle?

(40) Suppose that the angle $\angle PAQ$ is a plane angle of the dihedral angle $p_1 \cup \ell \cup q_1$. If $B \in \ell$ and $B \neq A$, compare $m(\angle PAQ)$ and $m(\angle PBQ)$.

(41) Let $\angle ABC$ be a plane angle of the right dihedral angle $a_1 \cup \ell \cup c_1$ such that $AB = 10 = BC$. Suppose that D is a point of ℓ such that $D \neq B$ and $BD = 10$. What is the area-measure of $\triangle ACD$?

(42) The bisector of a dihedral angle is the closed half-plane whose edge is the edge of the dihedral angle and which contains a bisector of a plane angle of the dihedral angle. If a point in the bisector is 10 inches from the edge of a dihedral angle of 60°, how far is the point from each of the planes whose union contains the dihedral angle?

(43) Suppose that line ℓ intersects plane p and that plane q contains ℓ.

(a) If $\ell \perp p$, does it follow that $q \perp p$?

(b) If ℓ is oblique to p [that is, intersecting but not perpendicular], does it follow that q is not perpendicular to p?

(44) Given the three planes p, q, and r.

(a) If $p \parallel q$ and $p \perp r$, does it follow that $q \perp r$?

(b) If $p \perp r$ and $q \perp r$, does it follow that $p \parallel q$?

(c) If $p \parallel q$ and $q \parallel r$, does it follow that $p \parallel r$?

(d) If $p \perp q$ and $q \perp r$, does it follow that $p \perp r$?

(45) Consider the two dihedral angles $p_1 \cup \ell \cup q_1$ and $r_1 \cup m \cup s_1$. Let p and r be parallel and q and s be parallel.

(a) What can you say about ℓ and m?

(b) What can you say about the measures of the given dihedral angles?

(46) Given three parallel planes p_1, p_2, and p_3, line ℓ which intersects the planes in the points A_1, A_2, and A_3, respectively, and line m which intersects the planes in the points B_1, B_2, and B_3, respectively.

(a) If $\ell \parallel m$ and $A_1A_2 = A_2A_3$, does it follow that $B_1B_2 = B_2B_3$?

(b) If $\ell \parallel m$, $A_1A_2/A_1A_3 = 0.4$, and $B_2B_3 = 12$, compute B_1B_2.

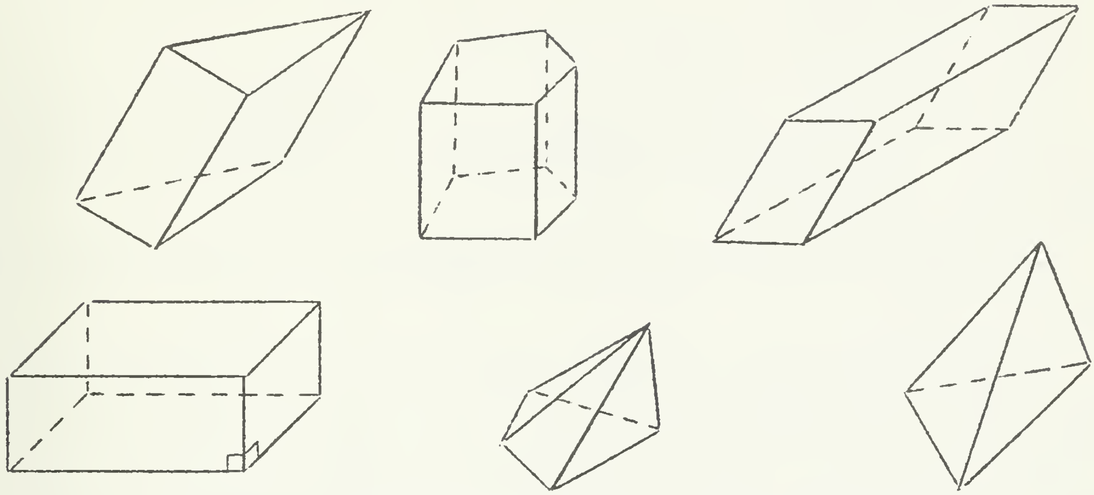
(c) If $A_1 = B_1$, $A_1A_2 = 3$, $A_2A_3 = 8$, and $B_1B_2 = 6$, compute B_1B_3.

(d) If $A_2 = B_2$, $A_1B_1 = 4$, $A_3B_3 = 12$, and $A_1A_3 = 8$, compute A_1A_2.

(47) In (46) let n be a third line which intersects the planes in the points C_1, C_2, and C_3, respectively. Suppose that $A_1 = B_1 = C_1$, A_2, B_2, and C_2, are three points, and $A_1A_2 = 1/2 \cdot A_1A_3$. If the area-measure of $\triangle A_2B_2C_2 = 7$, compute the area-measure of $\triangle A_3B_3C_3$.

SOME SIMPLE SOLIDS

Here are pictures of solids of some of the kinds we shall consider.



Four of the pictured solids are prisms and two are pyramids.

A prism may be thought of as a solid which is the union of parallel segments, all of the same length, connecting corresponding points of two congruent polygonal regions.

The two congruent polygonal regions are the bases of the prism, and the union of those parallel segments which join points of any two corresponding sides of the bases is a lateral face of the prism.

The union of the lateral faces of a prism is its lateral surface.

Each segment which joins corresponding vertices of the bases is a lateral edge of the prism.

The bases of a prism may be bounded by polygons of any kind, but the boundaries of the lateral faces are always parallelograms.

A triangular prism is one whose bases are triangles, a pentagonal prism is one whose bases are pentagons, etc.

A prism all of whose faces--bases, as well as lateral faces--are bounded by parallelograms is called a parallelepiped. [Any two opposite faces of a parallelpiped may be considered to be its bases.]

A prism which is the union of segments perpendicular to its bases is called a right prism. A rectangular parallelepiped is a right parallelepiped whose bases are rectangles. A cube is a rectangular parallelepiped whose faces are square regions.

The altitude of a prism is the [perpendicular] distance between its bases.

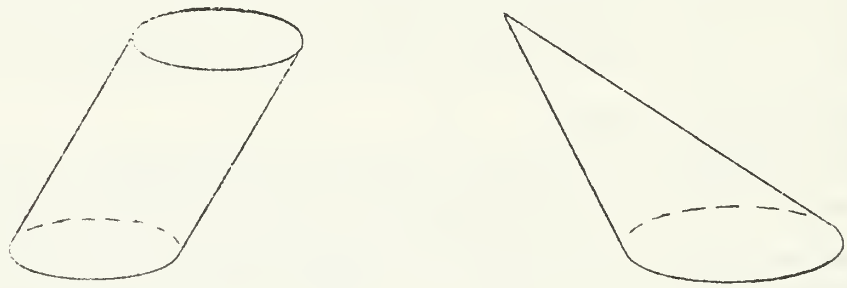
A pyramid is also a union of segments--the segments which join its vertex to the points of the polygonal region which is its base.

The lateral faces of any pyramid are triangular regions, but its base may be a polygonal region of any kind.

A pyramid with a 4-sided base is called a quadrangular pyramid. One whose base is a triangular region is called a triangular pyramid. [sometimes: a tetrahedron]. [Any face of a tetrahedron may be considered to be its base.]

The altitude of a pyramid is the [perpendicular] distance from its vertex to its base.

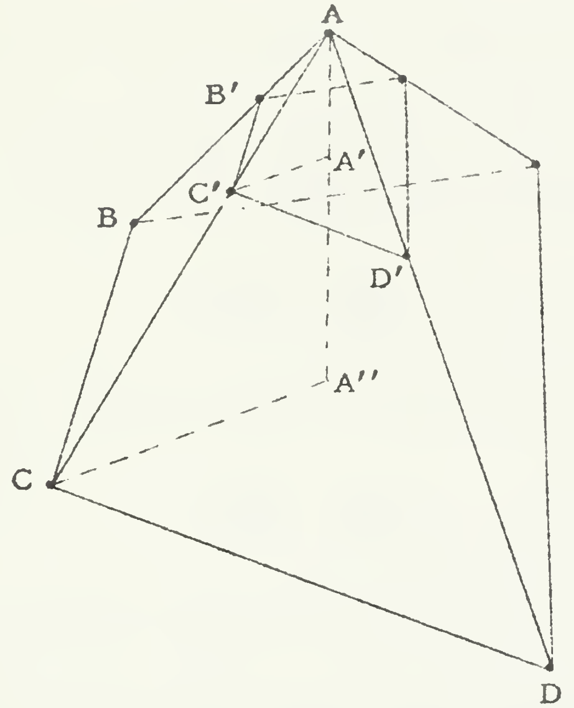
A regular pyramid is one whose base is a regular polygonal region whose center is the foot of the perpendicular from its vertex to its base. The lateral faces of a regular pyramid are bounded by congruent isosceles triangles, and its slant height is the common altitude of these triangles.



Cylinders and cones are analogous to prisms and pyramids, respectively. We shall consider only circular cylinders and cones--those whose bases are circular regions. Among these we single out the right circular cylinders--those which are unions of segments perpendicular to their bases--and the right circular cones--those for which the feet of the perpendiculars from their vertices to their bases are the centers of their bases. [These right solids can be pictured as being generated by revolving a rectangular region about one of its sides or a right triangular region about one of its legs.]

If a plane intersects a solid then their intersection is called a cross-section of the solid. Any solid is the union of its cross-section by all those planes parallel to any given plane which intersect the solid. When we speak of a cross-section of a prism, pyramid, cylinder or cone, we shall mean a cross-section by a plane parallel to the bases [or to the base]. We shall assume, as obvious, that all such cross-sections of a

prism or cylinder are congruent and, so, have the same area-measure. Since it is less obvious, we shall indicate a proof that all such cross-sections of a pyramid or cone are similar, and shall compute the ratio of similitude of such a cross-section to the base of the solid.



Consider any pyramid with vertex A and base bounded by a polygon BCD..., and a cross-section of this pyramid by some plane p′ parallel to the plane p which contains its base. The cross-section is bounded by a polygon B′C′D′... . Let A′′ be the foot of the perpendicular from A to p, and let A′ be the point of intersection of this perpendicular with p′. Then $\overline{AA''}$ is the altitude of the given pyramid and AA′ is the distance from A to p′. We shall show that the polygon B′C′D′... is similar to the polygon BCD..., and that the ratio of similitude is AA′/AA′′.

Consider any two adjacent vertices of the base of the pyramid--say B and C--and the plane which contains these points and the vertex A of the pyramid. Both the plane p and the plane containing A, B, and C contain B and C. So, the intersection of these two planes is the line $\overleftrightarrow{BC}$. Similarly, the line $\overleftrightarrow{B'C'}$ is the intersection of the plane p′ and the plane containing A, B, and C. Since $\overleftrightarrow{B'C'} \subseteq p'$ and $\overleftrightarrow{BC} \subseteq p$ and since [because p and p′ are parallel] $p' \cap p = \emptyset$, $\overleftrightarrow{B'C'} \cap \overleftrightarrow{BC} = \emptyset$. So, since $\overleftrightarrow{B'C'}$ and $\overleftrightarrow{BC}$ are coplanar [because both are subsets of the plane containing A, B, and C], it follows by definition that $\overleftrightarrow{B'C'} \parallel \overleftrightarrow{BC}$.

It follows from this that [in the plane containing A, B, and C] the matching $AB'C' \longleftrightarrow ABC$ is a similarity and, in particular, that

$$\frac{B'C'}{BC} = \frac{AC'}{AC}.$$

Now, consider the plane which contains A, A'', and C. [For the moment, we assume that, as in the figure, $A'' \neq C$.] Since $\overleftrightarrow{A'C'}$ and $\overleftrightarrow{A''C}$ are the intersections of this plane with the parallel planes p' and p, respectively, it follows [as above] that $\overleftrightarrow{A'C'} \parallel \overleftrightarrow{A''C}$. So [as above], the matching $AA'C' \longleftrightarrow AA''C$ is a similarity and, in particular,

$$\frac{AC'}{AC} = \frac{AA'}{AA''}.$$

[Of course, if, as we assumed was not the case, $A'' = C$, then $A' = C'$ and, trivially, $AC'/AC = AA'/AA''$.] Hence [in any case],

$$\frac{B'C'}{BC} = \frac{AA'}{AA''}.$$

Since B and C were any adjacent vertices of the base of the given pyramid, we have now shown that the sides of the boundary of the cross-section are proportional to the corresponding sides of the boundary of the base, and that the factor of proportionality is AA'/AA''.

To complete the proof that the polygons $B'C'D'\ldots$ and $BCD\ldots$ are similar [and that the ratio of similitude is AA'/AA''], it remains to be shown that corresponding angles of the two polygons are congruent. To do this, consider any corresponding vertices--say, C and C'. It is easy to see that the rays $\overrightarrow{C'B'}$ and $\overrightarrow{CB}$ are similarly directed [they are, as we have proved, parallel, and B' and B are on the same side of $\overleftrightarrow{C'C}$]. Likewise, $\overrightarrow{C'D'}$ and $\overrightarrow{CD}$ are similarly directed. Now, if $\angle BCD$ and $\angle B'C'D'$ were [as they aren't] in the same plane, it would follow from a theorem you proved in Unit 6 [Theorem 5-13 on page 6-154] that $\angle BCD$ and $\angle B'C'D'$ are congruent. Fortunately, Theorem 5-13 still holds for angles in space --if the sides of two angles can be matched in such a way that corresponding sides are similarly directed then the angles are congruent. So, $\angle B'C'D' \cong \angle BCD$ and, since C' and C were any corresponding vertices of the two polygons, we have proved that corresponding angles of the two polygons are congruent. Taken together with the fact already established that corresponding sides are proportional in the ratio AA'/AA'',

this shows that the polygons are similar and that the ratio of similitude is AA′/AA″.

[The "space-form" of Theorem 5-13 which we have appealed to follows from the "plane-form" and the fact that the polygons bounding the two bases of a right prism are congruent. If you wish, you can show how it follows, by considering the right prism one of whose bases is the cross section we have been considering and whose other base is in the plane p.]

The result we have established concerning cross-sections of pyramids and the fact that the ratio of the area-measures of two similar polygonal regions is the square of the ratio of similitude, yield the following result:

The ratio of the area-measure of a cross-section at a distance d from the vertex of a pyramid of altitude h to the area-measure of the base of the pyramid is d^2/h^2.

In particular:

Given any two pyramids which have the same altitude and whose bases have the same area-measure, it follows that the area-measure of any two cross-sections which are at the same distance from the bases are the same.

Since all cross-sections of a prism [because they are congruent] have the same area-measure, a statement similar to the last holds for prisms. The corresponding statements about cones and cylinders should now seem obvious enough to require no further comment.

EXERCISES

A. Makes sketches of the solids which fit these descriptions.

1. The three lateral faces are triangles and the base is a triangle.

2. The lateral faces are rectangles and a base is a triangle.

3. The edges are congruent and both bases are parallelograms.

4. A square pyramid whose lateral faces are equilateral triangles.

B. 1. Compute the area-measure of the total surface of a triangular pyramid whose lateral faces are equilateral triangles, given that one of the edges has measure e.

2. Repeat Exercise 1 given that the slant height is s.

3. A lateral edge of a right pentagonal prism has measure x and the perimeter of a base is y. Compute the total area-measure of its lateral faces.

4. The slant height of a pyramid is s and its base has perimeter p and area-measure K. Compute the area-measure of the total surface of the pyramid.

5. Suppose that a plane perpendicular to one of the lateral edges of a hexagonal prism intersects each lateral edge of the prism, and that the perimeter of the resulting cross-section is p. If S is the total of the area-measures of the lateral faces, compute the measure of a lateral edge.

6. Let d be the diameter of a base of a right circular cylinder. If the altitude of the cylinder is h, what is the area-measure of the total surface of the cylinder?

7. If you double the dimensions of a cube, what change takes place in the area-measure of the total surface of the cube?

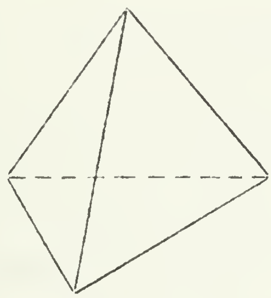
8. Suppose that the altitude of a regular quadrangular pyramid is x and each edge of the base has measure y. Compute the total area-measure of the lateral faces of the pyramid. [What change takes place in the total if the dimensions are tripled?]

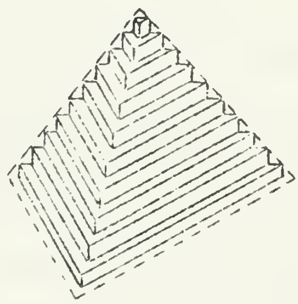
9. Suppose that the outside diameter of a 10 inch length of pipe is 3 inches. If the pipe is a quarter of an inch thick, what is the area of the total surface of the pipe [inside, outside, top, and bottom]?

10. A point-source of light is 10 feet from a vertical wall. If a circular piece of cardboard is held parallel to the wall and between the light and the wall, it casts a circular shadow on the wall. If the cardboard is 1 foot in diameter and held 2 feet from the point-source, what is the area of the resulting shadow?

AN AXIOM ON VOLUME-MEASURES

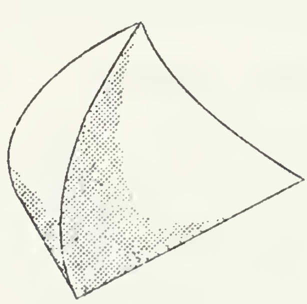
As you may know, one way to make a model of a solid--say, a triangular pyramid--is to begin by drawing pictures, on a thick sheet of cardboard, of cross-sections of the solid by planes which are parallel to a given plane and which are distant from one another [consecutively] by the thickness of the cardboard. Next, one cuts out these pictures and





pastes them together, one on top of the other. [The figures show a triangular pyramid and a model constructed in this way.] Finally, one fills in the "grooves" with plaster of Paris. Notice that the volume of the stack of pieces of cardboard is very nearly the volume of the finished model. In fact, no matter how short you are of plaster of Paris [as long as you have <u>some</u>] you can get by just by choosing thin enough cardboard.

One difficulty with making such a model is in properly centering each piece of cardboard over the preceding. And, if someone bumps into your stack of cardboard before the glue dries, you may end up with a model of a solid like the one pictured below.



However, whatever happens, the volume of cardboard in your model will stay the same.

Such considerations suggest that two solids--for example, the two pictured above--have the same volume if there is a plane p such that

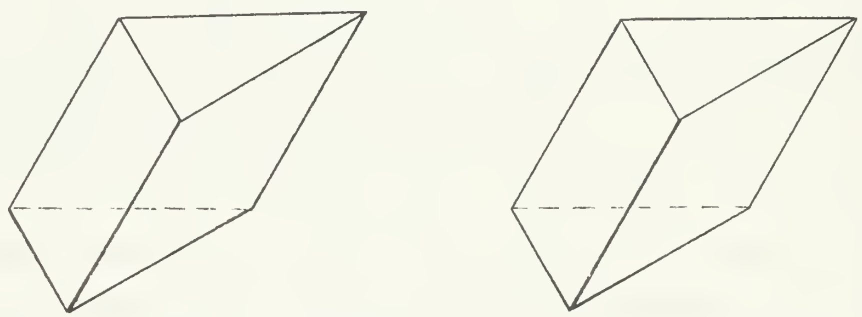
each plane parallel to p which intersects either of the solids also intersects the other and that the cross-sections of the solids by any such plane are congruent. This is, in fact, the case. Although we shall want a more comprehensive result, we can use this one to prove that

> two prisms which have the same altitude and whose bases are congruent have the same volume-measure,

and that

> two pyramids which have the same altitude and whose bases are congruent have the same volume-measure.

The first is easy to prove. Since we have agreed that we can "move" solids around without changing their volume-measures, we may assume that the two prisms have bases in the same plane, p [and, except for their bases, are on the same side of p]. Since the prisms have the same altitude, it follows that each plane parallel to p which intersects



either prism also intersects the other. Since all cross sections of a prism are congruent to the base, and since the bases of the two prisms are congruent, it follows that the cross-sections of the two prisms by any such plane [or by p itself] are congruent. Consequently, the two prisms have the same volume-measure.

The corresponding statement about pyramids is not much harder to prove. Again, we may assume that the two pyramids [with the same altitude, h, and congruent bases] have their bases in the same plane, p, and are otherwise "above" p. A plane parallel to p and at a distance $d < h$ above it cuts out cross-sections of the pyramids which are similar to the respective bases, the ratio of similitude being the same in both cases. So, since the bases are congruent, so are the "corresponding" cross-sections. Hence, the two pyramids have the same volume-measure.

In the same way one can argue that two cylinders [or two cones] which have the same altitude and congruent bases have the same volume-measure.

To obtain more complete results, let's return to the consideration of cardboard models and guess at a more comprehensive principle than the one we have been using about congruent cross-sections. This time, suppose that we have two solids such that each plane which is parallel to a given plane p and intersects either of the solids also intersects the other and that the two cross-sections of the solids by any such plane have the same area-measure. Suppose now, we picture, on cardboard, cross-sections of both solids by planes parallel to p and use them, as before, to build models of both solids. Each model will contain the same number of pieces of cardboard and corresponding pieces will have the same area and, of course, the same thickness--the thickness of the cardboard we choose to use. Consequently, it is natural to assume that corresponding pieces in the two models have the same volume--from which it follows that the two models have the same volume. Such considerations suggest:

> <u>Cavalieri's</u> <u>Principle</u>.
>
> Two solids have the same volume-measure if there is a plane p such that each plane which is parallel to p and intersects either of the solids also intersects the other, and the cross-sections of the solids by any such plane have the same area-measure.

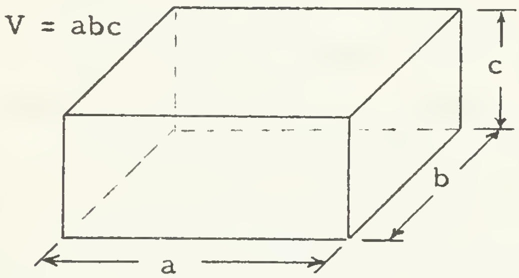
Using this principle [as an axiom] together with the fact that cross-sections of prisms and cylinders have the same area-measures as their bases, we can prove that

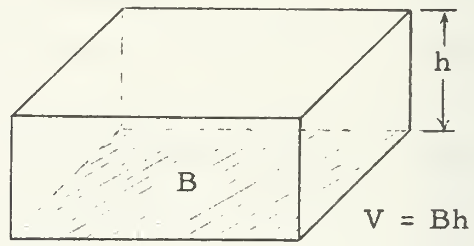
> two solids, each of which is either a prism or a cylinder, which have the same height and whose bases have the same area-measure, have the same volume-measure.

The proof is much like that of the theorem concerning prisms with the same height and congruent bases. A similar proof establishes the statement with 'prism or cylinder' replaced by 'pyramid or cone'.

VOLUME FORMULAS

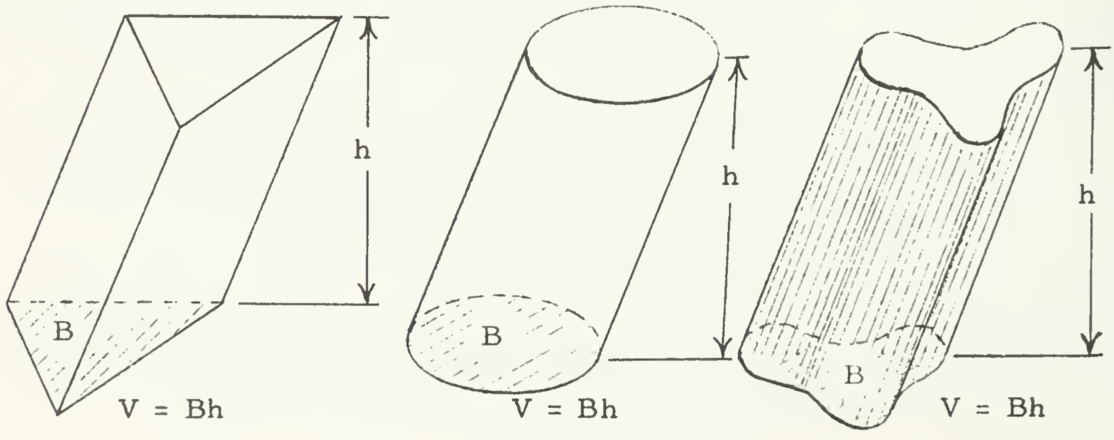
Cavalieri's principle gives us a sufficient condition in order that two solids have the same volume-measure. To obtain formulas for the volume-measures of solids we still need some place to start from. We shall choose as a starting place [as a second axiom] the formula for the volume-measure of a particular kind of prism--a rectangular parallelepiped.





Since the base of this prism is a rectangle with side-measures a and b, the area-measure of its base is ab. Since the altitude of the prism is c, the formula tells us that the volume-measure of a rectangular parallelepiped is the product of the area-measure of its base by its altitude.

We can now find a formula for the volume-measure of <u>any</u> prism or cylinder. For, consider any such solid whose base has area-measure B and whose altitude is h, and consider any rectangular parallelpiped whose base also has area-measure B and whose altitude is h. Using Cavalieri's principle, we have just proved that these two solids have the same volume-measure. Since that of the second is, by definition, Bh, that of the first is also Bh.



In particular, the volume-measure of a circular cylinder whose base has radius r and whose altitude is h is $\pi r^2 h$ [in terms of the diameter of the base, $V = \pi d^2 h/4$].

Having obtained pretty satisfactory results as to the volume-measures of prisms and cylinders, we now turn our attention to pyramids and cones. Evidently, we can parallel our work on volume formulas of prisms and cylinders once we find a formula for the volume-measure of one kind of pyramid. Triangular pyramids turn out to be easiest. We shall find that the volume-measure of a triangular pyramid is one third that of a prism which has the same base and altitude.

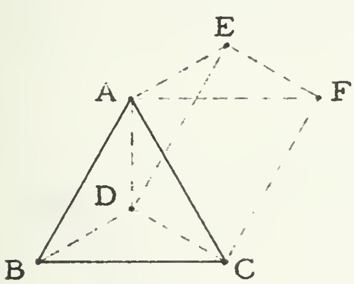


Fig. 1

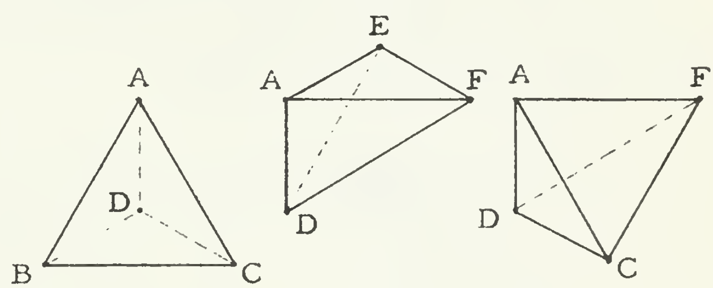


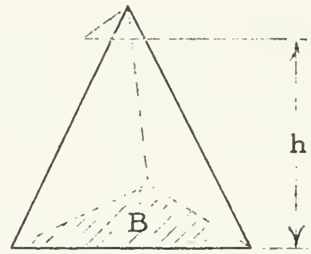
Fig. 2

Any triangular pyramid with vertex A and base $\triangle$BCD can be "enlarged" [as shown in Fig. 1] to a triangular prism which has the same altitude and base [$\overline{DE}$ and $\overline{CF}$ are parallel to $\overline{BA}$, and DE = BA = CF].

As shown in Fig. 2, this triangular prism is the union of three triangular pyramids. One way of seeing this is to notice, in Fig. 1, that the prism is the union of the given triangular pyramid and a pyramid whose vertex is A and whose base is ▱ CDEF. This latter quadrangular pyramid is cut into two triangular ones by the plane which contains A, D, and F. Notice that, since $\triangle CDF \cong \triangle EFD$, these two triangular prisms have congruent bases. Since the altitude of each is the distance between A and the plane containing ▱ CDEF, these two triangular prisms have the same volume-measure.

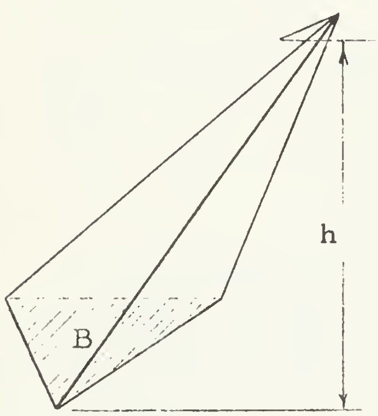
Consider, now, the first two pyramids pictured in Fig.2--the original pyramid with vertex A and base $\triangle$BCD and the pyramid with vertex D and base $\triangle$AEF. The bases of these pyramids are the two bases of the prism and, hence, are congruent. The altitude of each is merely the distance between the parallel planes which contain the bases of the prism. So, these two triangular pyramids have the same volume-measure.

Consequently, all three triangular pyramids have the same volume-measure--which must, then, be one third that of the prism.

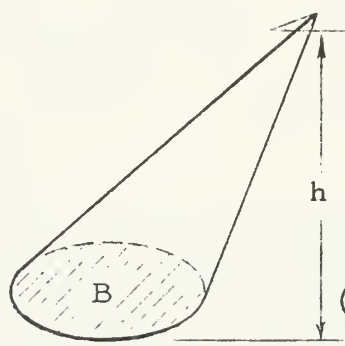


$$V = \frac{1}{3}Bh$$

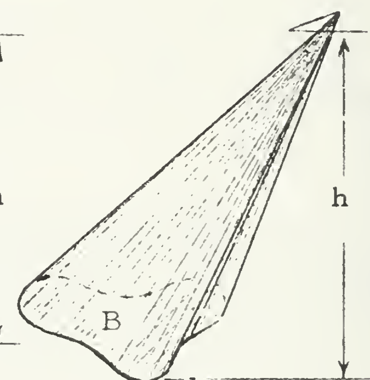
As in the case of prisms and cylinders, it now follows that the volume-measure of any pyramid or cone whose base has area-measure B and whose altitude is h is (1/3)Bh.



$$V = \frac{1}{3}Bh$$



$$V = \frac{1}{3}Bh$$



$$V = \frac{1}{3}Bh$$

In particular, the volume-measure of a circular cone whose base has radius r and whose altitude is h is $(1/3)\pi r^2 h$ [in terms of the diameter of the base, $V = \pi d^2 h/12$].

EXERCISES

A. Solve these problems.

1. What is the volume of a 14-foot high right circular cylinder with a 6-foot diameter?

2. Compute the volume of a parallelpiped whose altitude is 10 and which has a rectangular base with side-measures 3 and 7.

3. [The <u>axis</u> of a circular cylinder is the line determined by the centers of the bases. If the cylinder is not a right cylinder, the projection of the axis on the plane of either base is a line. The union of the axis and its projection contains four angles, two acute and two obtuse. The common measure of the acute angles is called <u>the inclination of the axis.</u>] The axis of a circular cylinder is inclined 30°. If the radius is 3 and the distance between centers of the bases is 10, what is the volume?

4. Suppose that the total area-measure of the six faces of a cube is the same number as the volume-measure. What is the measure of an edge of the cube?

5. If the volume of a circular cylinder is 180π cubic inches and the altitude (in inches) is 5, what is the diameter (in inches)?

6. Suppose that the volume of a right trapezoidal prism is 26 cubic inches and its altitude is 16 inches long. Find the distance (in inches) between the two parallel lateral faces if their areas are 168 square inches and 192 square inches, respectively.

7. The common length of the four diagonals of a rectangular solid [that is, a right rectangular prism] is 13 inches and two of the edges are 7 inches and 8 inches, respectively. Compute the volume of the solid.

8. The diagonal of a cube [that is, the measure of any of the four segments which join vertices but are not contained in any of the faces] is $4\sqrt{3}$. Compute the volume-measure of this cube.

9. The dimensions of a rectangular solid are in the ratio 2:3:4 and the diagonal is 58. Compute the volume-measure of this solid.

10. If you double the dimensions of a rectangular solid, what change takes place in the volume?

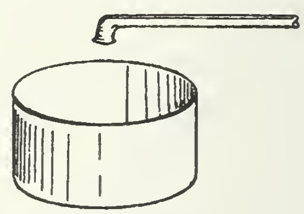
11. If the area-measure of the total surface of a cube is t, what is the volume-measure?

12. If the diagonal of a face of a cube is d, what is the volume-measure?

13. Find the volume-measure of the largest right circular cylinder which is contained in a cube of side-measure e.

14. Find the volume-measure of the largest right prism with a right-triangular base which is contained in a right circular cylinder whose radius is 3 and whose altitude is 10.

15. A water tank in the shape of a right circular cylinder has a diameter of d feet. How long will it take to fill the tank to a depth of x feet if the water flows into the tank through a pipe of inside diameter y inches at the rate of r feet per minute?



B. Solve these problems.

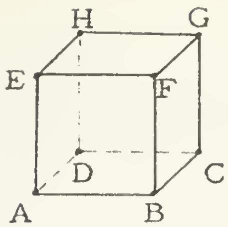
1. A regular square pyramid is 3 inches tall and the perimeter of its base is 16 inches. What is its volume?

2. A regular triangular pyramid has equilateral-triangular faces. If each edge of the pyramid is 10 inches long, what is its volume?

3. How tall is a pyramid whose volume is 21 cubic inches if its base is bounded by a rhombus with diagonals 6 inches and 7 inches, respectively?

4. What is the volume-measure of the largest circular cone contained in a circular cylinder whose volume-measure is V?

5. A cross-section of a pyramid 10 inches from its vertex has an area of 100 square inches. If the base of the pyramid has an area of 200 square inches, compute the volume of the pyramid.

6. 

The diagram at the left is a picture of a cube with edge-measure e. Compute the volume - measure of each of the pyramids described in the following exercises.

(a) vertex F and base $\square$ABCD

(b) vertex F and base $\triangle$HGD

(c) vertex P and base $\square$ABCD where $\{P\} = \overleftrightarrow{EG} \cap \overleftrightarrow{HF}$

(d) vertex Q and base $\square$ABCD where $\{Q\} = \overleftrightarrow{AG} \cap \overleftrightarrow{EC}$

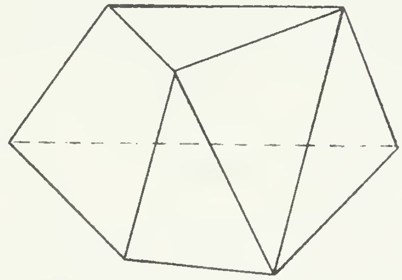
(e) vertex A and base ▭HFBD

(f) vertex E and base ▭HABG

(g) vertex F and base $\triangle$HGA

PRISMATOIDS

A prismatoid is a solid which, like a prism, has for its bases two polygonal regions in parallel planes. The prismatoid is the union of all segments which have an end point in each base. Its lateral faces are triangular or quadrangular regions whose vertices are vertices of the



bases. The quadrangular faces are either trapezoidal or, in special cases, bounded by parallelograms. Each prism is a prismatoid, and pyramids can be considered limiting cases of prismatoids--when one base is "shrunk" to a point.

The midsection of a prismatoid is the section made by a plane midway between the planes which contain its bases. The altitude of a prismatoid is the distance between the planes which contain its bases.

Despite the wide variety of prismatoids, there is a simple formula

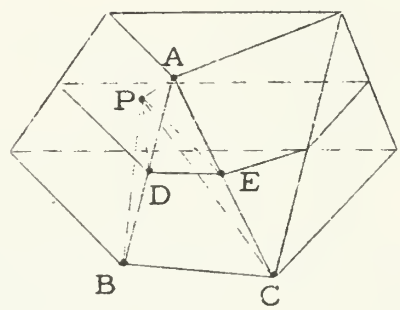
for the volume-measure of any of them.

If the area-measures of the bases of a prismatoid are B_1 and B_2, that of its midsection is M, and its altitude is h, then its volume-measure V is given by:

$$V = \frac{1}{6}(B_1 + B_2 + 4M)h$$

This formula is called the <u>prismoidal</u> [or: prismatoidal] <u>formula</u>. Notice that, as it should, it gives the correct result for a prism [$B_1 = B_2 = M = B$] and for a pyramid [$B_1 = B$, $B_2 = 0$, $M = B/4$].

The prismoidal formula can be derived by investigating the result of cutting a prismoid up into triangular pyramids. Here's how.



Choose a point P in the midsection of a prismatoid. The prismatoid is, then, the union of the pyramids which have P as vertex and, for bases, the faces of the prismatoid. Two of these, those whose bases are the bases of the prismatoid, have easily computed volume-measures. These measures are

$$\frac{1}{3}B_1\left(\frac{h}{2}\right) \quad \text{and} \quad \frac{1}{3}B_2\left(\frac{h}{2}\right)$$

respectively--that is,

$$\frac{1}{6}B_1h \quad \text{and} \quad \frac{1}{6}B_2h.$$

We shall see that the volume-measures of the other pyramids--those whose bases are lateral faces of the prismatoid--add up to

$$\frac{2}{3}Mh,$$

thus completing the proof of the prismoidal formula.

Since some lateral faces of a prismatoid may be trapezoidal and others triangular, some of the pyramids we have left to consider may

have trapezoidal bases and others may have triangular bases. To simplify the argument we note that each pyramid with a trapezoidal base is split by a plane through its vertex and a diagonal of its base into two triangular pyramids. So, the pyramids whose volume-measures we have left to compute may be taken to be triangular pyramids with vertex P whose bases are either lateral faces of the prismatoid or "halves" of lateral faces. Consider such a pyramid, with base $\triangle ABC$. The midsection of the prismatoid intersects $\triangle ABC$ in a segment $\overleftrightarrow{DE}$ parallel to $\overleftrightarrow{BC}$, and $K(\triangle ADE) = \frac{1}{4} \cdot K(\triangle ABC)$. Consequently, the volume-measure of the pyramid with vertex P and base $\triangle ABC$ is 4 times that of the pyramid with vertex P and base $\triangle ADE$. Since this latter pyramid may also be described as having vertex A and base $\triangle PDE$, its volume-measure is

$$\frac{1}{3} \cdot K(\triangle PDE)\frac{h}{2}.$$

So, the volume-measure of the pyramid with vertex P and base $\triangle ABC$ is

$$\frac{2}{3} \cdot K(\triangle PDE)h.$$

In other words, the volume-measure of each of the pyramids we are considering--those with vertex P and a lateral face, or "half" a lateral face, for base--is the product of two thirds the area-measures of its intersection with the midsection of the prismatoid by the altitude h of the prismatoid. Consequently, as we set out to show, the sum of the volume-measures of those pyramids is

$$\frac{2}{3}Mh.$$

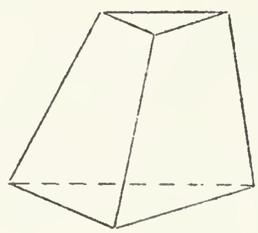
Since $\frac{1}{6}B_1h + \frac{1}{6}B_2h + \frac{2}{3}Mh = \frac{1}{6}(B_1 + B_2 + 4M)h$, we have established the prismoidal formula.

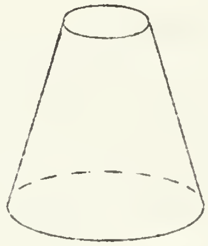
Note that we have assumed that the midsection is of such a shape that it is the union of nonoverlapping triangular regions with vertex P. This is the case for any choice of P in the midsection if the latter is convex--the case usually treated.

The prismoidal formula also works for solids which are like prismatoids but whose bases, in parallel planes, are not polygonal regions --that is for solids which are unions of segments which have one end point in each of two parallel plane regions.

FRUSTUMS

A frustum of a pyramid [or cone] is a solid consisting of the points of the pyramid [or cone] between or on the base of the pyramid [or cone] and a cross-section parallel to the base. A frustum of a pyramid is a prismatoid, and the prismoidal formula applies to frustums of both pyramids and cones.





In the case of a frustum one can express the area-measure M of the midsection rather simply in terms of the area-measures of the bases, and, so, obtain a useful special case of the prismoidal formula. To do so, suppose that the area-measure of the base of the pyramid [or cone], whose frustum we are considering, is B_1, the area-measure of the other base of the frustum is B_2 and that that of the midsection of the frustum is M. Suppose, also, that the altitude of the frustum is h and that that of the pyramid [or cone] is h_1. It follows that the "upper" base of the frustum is at a distance $h_1 - h$ from the vertex of the pyramid [or cone], and that the midsection is at a distance $h_1 - \frac{h}{2}$ from the vertex. Consequently,

$$B_2 = \left(\frac{h_1 - h}{h_1}\right)^2 B_1 \quad \text{and} \quad M = \left(\frac{h_1 - h/2}{h_1}\right)^2 B_1$$

$$= \left(1 - \frac{h}{h_1}\right)^2 B_1 \qquad\qquad = \left(1 - \frac{h}{2h_1}\right)^2 B_1$$

It follows that $1 - \frac{h}{h_1} = \sqrt{B_2/B_1}$ and, so, that $\frac{h}{h_1} = 1 - \sqrt{B_2/B_1}$.

Consequently,

$$\frac{h}{2h_1} = \frac{1 - \sqrt{B_2/B_1}}{2},$$

$$1 - \frac{h}{2h_1} = 1 - \frac{1 - \sqrt{B_2/B_1}}{2}$$

$$= \frac{1 + \sqrt{B_2/B_1}}{2},$$

and, so,

$$\left(1 - \frac{h}{2h_1}\right)^2 = \frac{1 + 2\sqrt{B_2/B_1} + B_2/B_1}{4} .$$

Hence,
$$M = \frac{1 + 2\sqrt{B_2/B_1} + B_2/B_1}{4} \cdot B_1$$

$$= \frac{B_1 + 2\sqrt{B_1B_2} + B_2}{4} .$$

Substituting into the prismoidal formula, one obtains, for the volume-measure V of a frustum of a pyramid [or cone], the formula:

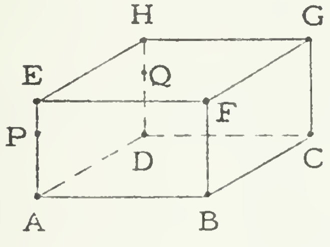
$$V = \frac{1}{3}(B_1 + B_2 + \sqrt{B_1B_2})h$$

Notice that the formula checks with earlier formulas in the case where the frustum is the entire pyramid [or cone].

EXERCISES

Solve these problems.

1. The altitude of a frustum of a regular square pyramid is 12 and the area-measures of the bases, 36 and 144, respectively. Compute the volume-measure of this frustum.

2. 

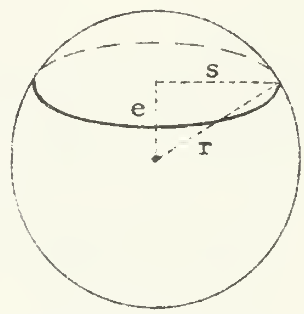
The diagram at the left pictures a rectangular solid such that AB = 14, BC = 10, and FB = 8. Let P and Q be points of $\overline{EA}$ and of $\overline{HD}$, respectively, such that AP = DQ. Consider the plane containing the points F, G, Q, and P. Determine AP so that this plane separates the rectangular solid into two solids whose volume-measures are in the ratio 1 : 2.

3. Use the same rectangular solid described in Exercise 2, but this time let AP = 7, DQ = 5, and let R be the point of CG such that CR = 3. Find the volume-measure of the portion of the rectangular solid which is "under" the union of the planes FPQ and FRQ.

SOLID SPHERES

We shall now find a formula for the volume-measure of a spherical region. [A spherical region is one which is bounded by a sphere. Such a solid is sometimes confusingly called 'a sphere'. Better short names are 'solid sphere' and 'ball'.] As you might expect, we shall use Cavalieri's principle.

Consider a sphere of a radius r and a cross-section [of the solid sphere which it bounds] made by a plane at a distance $e < r$ from the center of the sphere. This cross-section is a circular region whose

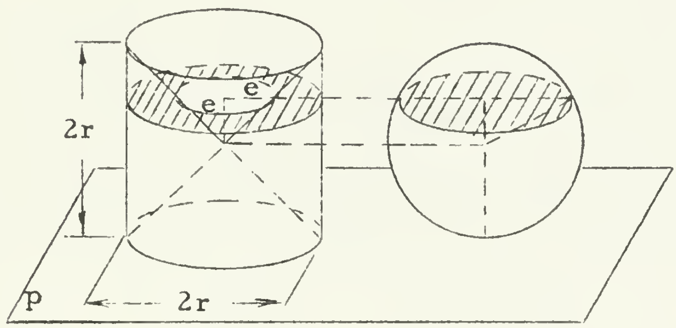


radius, s, is $\sqrt{r^2 - e^2}$ and whose area-measure is, consequently, $\pi(r^2 - e^2)$. At this point, one needs an inspiration. What solid whose volume-measure we already know how to find has cross-sections whose area-measures are the same as those of these circular cross-sections? Struggling a bit, we note that

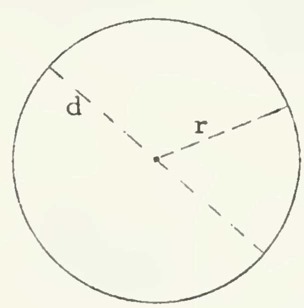
$$\pi(r^2 - e^2) = \pi r^2 - \pi e^2,$$

and that all the cross-sections of a cylinder of radius r have area-measure πr^2. Moreover, $0 \leq e \leq r$ and, for each such $e > 0$, πe^2 is the area-measure of a circular region of radius e. Thinking of the cylinder and Cavalieri's principle may suggest that we consider a right circular cylinder of radius r and of altitude $2r$--the "altitude" of the solid sphere. If we choose a point on the sphere and let p be the plane through this point and perpendicular to the corresponding diameter of the sphere [that is, the plane tangent to the sphere at the chosen point], and if we choose for the right circular cylinder one whose base is in the plane p, we have a set-up for exploiting Cavalieri's principle. Each plane parallel to p and at a distance $e \leq r$ above or below the centers of the sphere and cylinder cuts out cross-sections of the two solids. For the solid

sphere the cross-section is a circular region whose area-measure is $\pi r^2 - \pi e^2$ and, for the cylinder, it is a circular region whose area-measure is πr^2. Now, it is easy to find a circular region in the latter



cross-section whose area-measure is πe^2. Just take the one with the same center and radius e. Since the diameter and altitude of the cylinder are equal, this circular region is merely a cross-section of one of the two right circular cones whose common vertex is the center of the cylinder and whose bases are the bases of the cylinder. By Cavalieri's principle, the volume-measure of the spherical region is the same as that of the solid which remains when these two cones are "bored out" of the cylinder. But, we know the volume-measure of this solid. It is the difference between that of the cylinder, $\pi r^2 \cdot 2r$, and that of the two cones, $2(\frac{1}{3}\pi r^2 \cdot r)$. So, the volume-measure of the solid sphere is $2\pi r^3 - \frac{2}{3}\pi r^3$--that is, $\frac{4}{3}\pi r^3$.

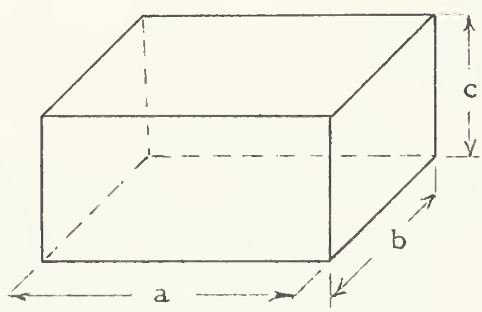


$$V = \frac{4}{3}\pi r^3 = \frac{\pi d^3}{6}$$

It is interesting to notice that the volume-measure of the spherical region is 2/3 that of the smallest right circular cylinder which could contain it.

SURFACE AREA FORMULAS

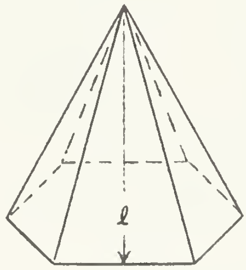
The area-measures of the lateral surfaces or the total surfaces [including bases] of prisms and pyramids are usually fairly easy to come by, but, except in quite simple cases, the formulas are difficult to state. In general, one computes the area-measures of the faces in question and adds the results. All the faces are polygonal regions, and most of them are bounded by either parallelograms or triangles. As an example which leads to a simple formula for the area-measure of the total surface, we cite the case of a rectangular parallelepiped.



$$S = 2(ab + bc + ca)$$

The only other example [among prisms and pyramids] worth bothering about is that of a regular pyramid--one whose base is bounded by a regular polygon whose center is the foot of the perpendicular from the vertex of the pyramid.

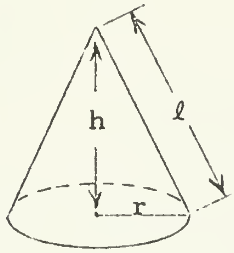
Since all the lateral faces are congruent [isosceles] triangles whose bases are the sides of the regular polygon and whose common altitude is the slant height of the pyramid, the area-measure L of the lateral surface is very easy to compute.



$L = \frac{1}{2}p\ell$, where p is the perimeter of the base and ℓ is the slant height

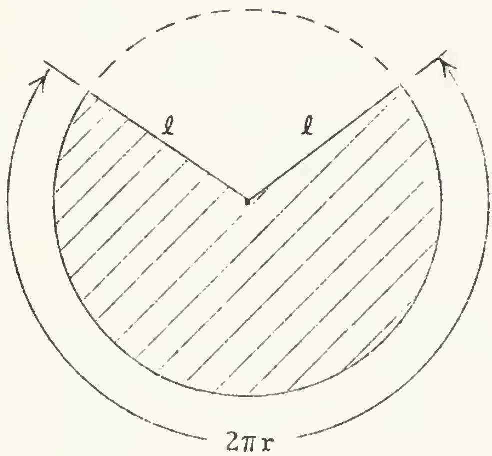
To obtain the area-measure S of the total surface, one adds to L the area-measure of the base.

This last result suggests the formula for the area-measure of the lateral surface of a right circular cone of radius r and slant height ℓ.



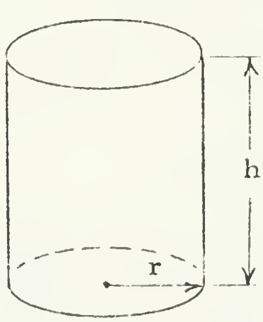
$$L = \pi r \ell = \pi r \sqrt{r^2 + h^2}$$
$$[\text{and, } S = \pi r(\ell + r).]$$

Further evidence for the correctness of this formula [but not a proof] can be obtained by thinking of the lateral surface as slit along one of the



segments joining the vertex to the base [like an Indian tepee] and flattened out. The result is a circular sector of radius ℓ and arc-measure $2\pi r$. And, as we know, the area-measure of such a sector is $\frac{1}{2}\ell \cdot 2\pi r$.

A similar procedure suggests formulas for right circular cylinders.



$$L = 2\pi rh$$
$$S = 2\pi r(h + r)$$

Somewhat similar formulas [$L = ph$ and $S = ph + 2b$] hold for all <u>right</u> prisms and cylinders.

Finally, we can arrive, in a somewhat backhanded way, at the formula for the area-measure of a sphere of radius r. Since the volume-measure of the region bounded by such a sphere is $\frac{4}{3}\pi r^3$, that of a spherical shell of outer radius r and thickness t must be

$$\frac{4}{3}\pi r^3 - \frac{4}{3}\pi(r - t)^3.$$

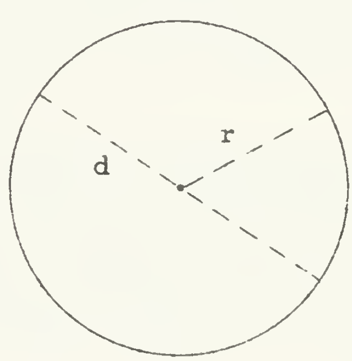
On the other hand, if t is small and S is the area-measure of the outer surface of the shell, then the same volume-measure should be approximately

$$St.$$

So,

$$S \doteq \frac{4}{3}\pi[r^3 - (r - t)^3]/t,$$

Since, for sufficiently small values of t, $3r^2 - 3rt + t^2$ is very nearly $3r^2$, we can guess that S is exactly $\frac{4}{3}\pi \cdot 3r^2$. This guess is correct [but, the preceding argument is not a proof].

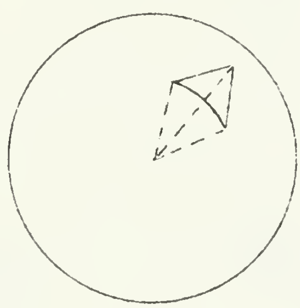


$$S = 4\pi r^2 = \pi d^2$$

So, the area-measure of the surface of a sphere is 4 times that of its largest cross-section. Recalling our comparison of the volume-measures of a spherical region and of the smallest right circular cylinder which contains it [the former was 2/3 of the latter], we see a rather

surprising similarity. The ratio of the area-measures of the total surfaces of the solids is $4\pi r^2/[2\pi r(2r + r)]$--again 2/3. This so pleased the Greek geometer, Archimedes, who first gave adequate proofs of these results, that he asked that a picture of a sphere and circumscribed cylinder be engraved on his tomb.

There is another way to make the formula for the area-measure of a sphere appear reasonable [and to help you to remember both this formula and that for the volume-measure of a solid sphere]. This is to think of a solid sphere as the union of a lot of narrow pyramid-like solids whose common vertex is the center of the sphere and whose "bases" are portions of the surface of the sphere. Since the bases are



curved rather than plane, these solids will not actually be pyramids. However, the volume-measure of each should be pretty close to one third the product of the area-measure of its "base" by its "altitude", the radius of the sphere. Consequently, the area-measure S of the sphere and the volume-measure V of the region it bounds may be expected to satisfy the equation:

$$V = \frac{1}{3}Sr$$

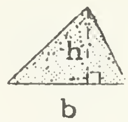
If you recall this and either one of the formulas:

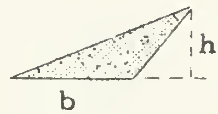
$$V = \frac{4}{3}\pi r^3 \qquad\qquad S = 4\pi r^2$$

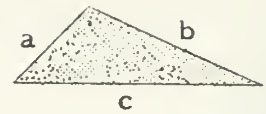
it is easy to derive the other.

SUMMARY OF MENSURATION FORMULAS

Triangular regions



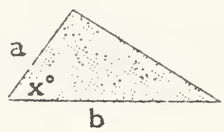


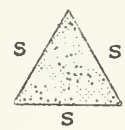


$K = \frac{1}{2}hb$

$K = \sqrt{s(s - a)(s - b)(s - c)}$

$[s = \frac{1}{2}(a + b + c)]$



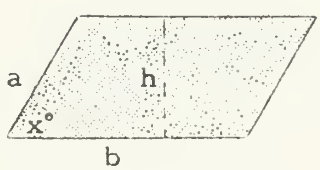


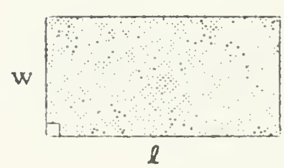
$K = \frac{1}{2}ab \sin x°, \quad [0 < x < 90]$

$K = \frac{1}{2}ab \sin (180 - x)°, \quad [90 < x < 180]$

$K = \frac{s^2}{4}\sqrt{3}$

Quadrilateral regions

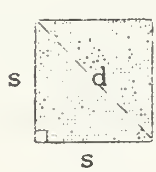


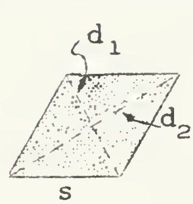


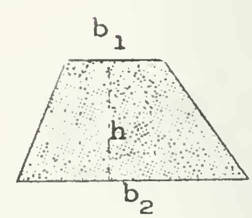
$K = hb = \frac{1}{2}ab \sin x°, \quad [0 < x < 90]$

$K \quad = \frac{1}{2}ab \sin (180 - x)°, \quad [90 < x < 180]$

$K = w\ell$



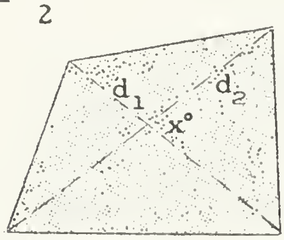




$K = s^2 = \frac{d^2}{2}$

$K = \frac{1}{2}d_1 d_2$

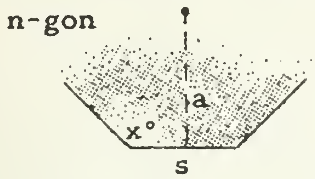
$K = \frac{1}{2}h(b_1 + b_2)$



$K = \frac{1}{2}d_1 d_2 \sin x° \quad [0 < x < 90]$

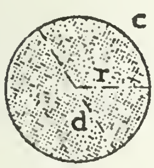
$K = \frac{1}{2}d_1 d_2 \sin (180 - x)° \quad [90 < x < 180]$

Regular polygonal regions

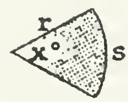


$$K = \frac{1}{2}ans = \frac{1}{2}ap$$
$$= \frac{1}{4}ns^2\tan\left(\frac{x}{2}\right)^\circ$$
$$= \frac{1}{4}ns^2\tan\left(\frac{(n-2)180}{2n}\right)^\circ$$

Circular regions

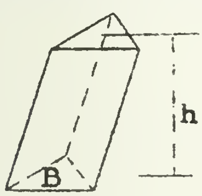


$$c = 2\pi r = \pi d$$
$$K = \pi r^2 = \frac{1}{4}\pi d^2$$

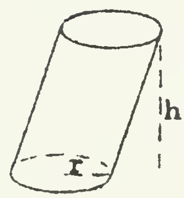


$$\frac{s}{c} = \frac{x}{360} \quad [0 < x < 180]$$
$$K = \frac{x}{360}(\pi r^2) \quad [0 < x < 180]$$

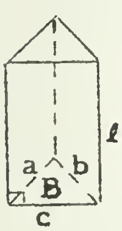
Prismatic solids



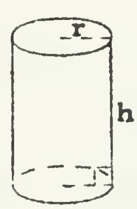
$$V = hb$$



$$V = \pi hr^2$$
$$= \frac{1}{4}\pi hd^2$$

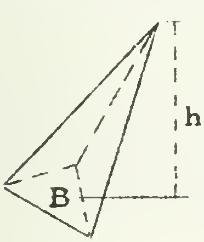


$$L = \ell(a+b+c)$$
$$S = L + 2B$$

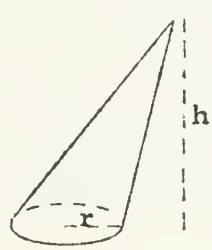


$$L = 2\pi hr = \pi hd$$
$$S = 2\pi r(h+r)$$

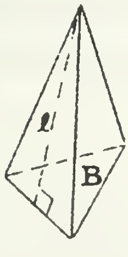
Pyramidal solids



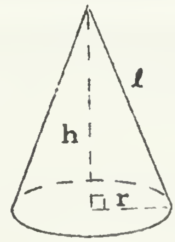
$$V = \frac{1}{3}hB$$



$$V = \frac{1}{3}\pi hr^2$$



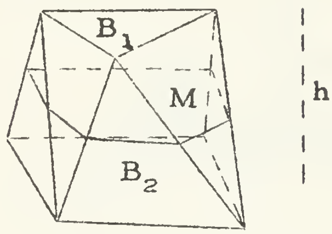
$$L = \frac{1}{2}\ell p$$
$$S = \frac{1}{2}\ell p + B$$



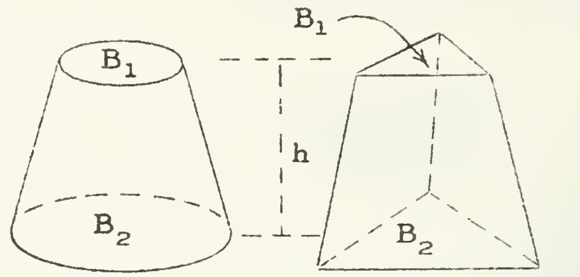
$$L = \pi\ell r = \pi r\sqrt{r^2 + h^2}$$
$$S = \pi r(\ell + r)$$

regular base, congruent faces

Prismatoidal solids



$$V = \frac{1}{6}h(B_1 + B_2 + 4M)$$

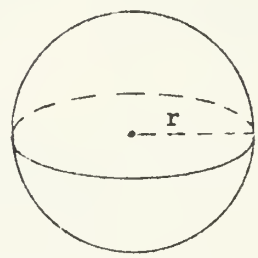


$$V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$$

Spherical solids

$$V = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$$

$$S = 4\pi r^2 = \pi d^2$$



EXERCISES

A. Solve these problems.

1. Compute the altitude of a right prism whose total surface has area-measure 320 and one of whose bases is a rhombus whose diagonal-measures are 6 and 10, respectively.

2. Suppose that the radius of the regular base of a right hexagonal prism is 6. If the altitude is 12, compute the area-measure of the total surface and compute the volume-measure.

3. Repeat Exercise 2, but triple the radius and double the altitude. [Compare the new area- and volume-measures with the ones obtained in Exercise 2.]

4. Compute the lateral area-measure of a right prism whose altitude is x and whose base is a right triangle with legs of measures 3x and 4x, respectively.

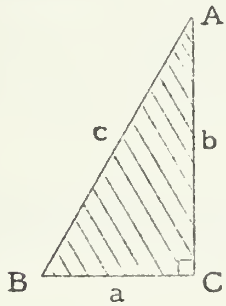
5. The edge-measures of two cubes are in the ratio $p:q$. Compute the ratio of the area-measures of their total surfaces and the ratio of their volume-measures.

6. Suppose that the radii of two circular cylinders are r_1 and r_2, respectively, and their altitudes are h_1 and h_2, respectively. Compute the ratios of their lateral area-measures and the ratio of their volume-measures.

7. If the ratio of the radii of two solid spheres is p to q, what are ratios of their surface area-measures and their volume-measures?

8. Compute the total area-measure of a regular hexagonal pyramid if the perimeter of the base is 24 and its altitude is 10.

9. Consider a right circular cone whose "apex angle" is an angle of 120°. If the radius of the base is 10, what are the lateral area-measure and the volume-measure?

10. 

Visualize the solid generated by rotating $\triangle ABC$ around $\overleftrightarrow{AC}$. Such a solid is called a <u>solid</u> <u>of</u> <u>revolution</u>. Of course, in this case, the solid is a right circular cone. Compute its volume-measure and its lateral area-measure.

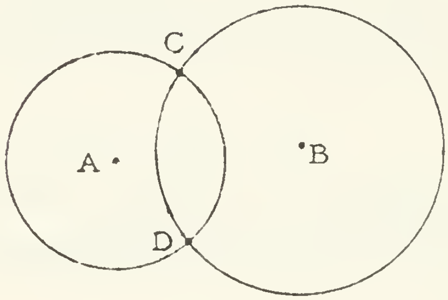
11. Repeat Exercise 10 for the solids of revolution obtained by rotating $\triangle ABC$ around $\overleftrightarrow{BC}$ and around $\overleftrightarrow{AB}$.

12. Compute the total area-measure and the volume-measure of the solid of revolution obtained by rotating a rectangular region of side-measures 24 and 32 around the line containing one of the shorter sides.

☆13. Repeat Exercise 12 for the solid of revolution obtained by rotating the region around the line containing a diagonal.

B. Solve these problems.

1. Consider a solid sphere with radius 10. Compute the area-measure of the intersection of the solid sphere and a plane whose distance from the center is 3.

2. 

Given the circular regions with centers A and B such that AC = 3, CB = 4, and BA = 5. Consider the solid of revolution obtained by rotating both regions around $\overleftrightarrow{AB}$. What is the area-measure of the region generated by $\overline{CD}$?

3. If 6 is the radius of the smallest solid sphere which contains a right circular cylinder of altitude 10, what is the volume-measure of the cylinder?

4. The intersection of a sphere and a plane containing its center is called <u>a great circle</u> of the sphere. If the area-measure of the region [interior to the sphere] bounded by a great circle is x, what are the area-measure of the surface of the solid sphere and the volume-measure?

5. Write a formula for the surface area-measure S of a sphere in terms of its volume-measure V.

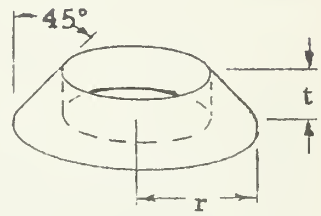
6. Compute the radius of the smallest solid sphere which contains the smallest cube which contains a solid sphere of radius r.

7. Two spherical balls of radii 7 and 10, respectively, are placed on a table so that they touch each other. How far apart are their points of contact with the table?

8. What is the volume-measure of the largest sphere which is contained in the frustum of a right circular cone whose bases have radii 4 and 16?

9. Compute the diameter of the smallest spherical solid which contains a rectangular solid with edge-measures x, y, and z.

10. Find a formula for the volume measure of the holed frustum, as pictured.



☆11. Consider the set of all rectangular regions of perimeter p. Which of these regions produces the solid of revolution of maximum volume when it is rotated around the line containing one of its sides? [Hint. Let w be the measure of the side contained in the axis of revolution. Find the function that V is of w. Graph the function and guess its maximum value.]

C. Each of the following exercises gives certain conditions. Describe the geometric figure which satisfies the given conditions.

Example 1. The set of points each of which belongs to a plane p and each of which is the distance d from the point $P \in p$.

Solution. The circle in p with center P and radius d.

Example 2. The set of points each of which is equidistant from the two points A and B.

Solution. The plane perpendicular to $\overleftrightarrow{AB}$ and which contains the midpoint of $\overline{AB}$.

1. The set of points each of which is the distance d from P.

2. The set of points each of which belongs to a plane p and is equidistant from the two points $A \in p$ and $B \in p$.

3. The set of points each of which is the distance d from a line ℓ.

4. The set of points each of which belongs to a plane p and is the distance d from a line $\ell \subseteq p$.

5. The locus of points [that is, the set of points] each of which is the distance d from a plane p.

6. The locus of points each of which is equidistant from two parallel planes p and q.

7. The locus of points each of which is equidistant from two intersecting planes p and q.

8. The locus of points each of which is twice as far from plane p as from plane q where $p \parallel q$.

9. The locus of points P such that, for some point X of the circle c of radius r, $PX = \frac{1}{3}r$.

10. Repeat Exercise 9 but add the condition that P be in the plane of c.

11. The locus of points each of which belongs to plane ABC and is equidistant from A, B, and C.

12. The locus of points each of which is equidistant from the vertices of $\triangle ABC$.

13. The locus of points each of which is equidistant from the parallel lines ℓ and m.

14. The locus of points each of which is equidistant from the noncoplanar parallel lines ℓ, m, and n.

15. The locus of points P such that $AP = d_1$, $BP = d_2$, and $A \neq B$. [Consider all cases.]

16. The locus of points P such that $AP = d$ and P is equidistant from the parallel planes p and q.

17. The locus of points P such that $PA = d$ and $PB = PC$.

APPENDIX E

[This Appendix contains proofs of Theorems 221-224. These theorems are the basis of the applications discussed in section 9.10, and the Appendix ends with a summary of the results needed for such applications.]

Some functional equations.--Many problems you have met in your study of mathematics require, for their solution, that one find numbers which satisfy given conditions. [Most of the "word problems" you have solved are of this kind.] Many such problems boil down to finding the solutions of an algebraic equation. Other problems--there are several examples in section 9.10--require one to find functions which satisfy given conditions. In many cases such problems boil down to solving a so-called functional equation. Here are two examples of functional equations:

$$(1) \qquad \forall_x\, f(x) = 2f(x)$$

$$(2) \qquad \forall_x \forall_y\, f(x) = f(y)$$

Both (1) and (2) are open sentences--neither true nor false. From either we can generate a statement by replacing 'f' by a name for some function. Some of the statements obtained in this way will be true and some will be false. [Some replacements will lead to nonsense--for example, '$\forall_x \sqrt{x} = 2\sqrt{x}$'--because the function in question is not defined for all real numbers.] Those functions for which (2), say, becomes a true statement are said to satisfy (2), or to be solutions of (2). To solve a functional equation is to determine which functions are its solutions.

The functional equations (1) and (2) are easy to solve. For example, (1) is equivalent to:

$$\forall_x\, f(x) = 0 \qquad \text{[Explain.]}$$

So, the only solution of (1) is the constant function 0. What are the solutions of (2)?

In this Appendix we shall study some functional equations which occur frequently in mathematics. As we shall see, it is often difficult to solve a functional equation completely--that is, to find all its solutions. However, it is often possible to find all the solutions of a given functional equation which have some "desirable" property, such as monotonicity or continuity, and to prove that all its remaining solutions have some "undesirable" property.

THE LDPMA AND THEOREMS 205, 215, AND 218

Among the functions we have studied, four kinds stand out:

(a) the homogeneous linear functions

$$[f = \{(x, y): y = cx\}, \text{ for some } c \neq 0]$$

(b) the exponential functions

$$[f = \{(x, y): y = a^x\}, \text{ for some } a > 0]$$

(c) the logarithm functions

$$[f = \{(x, y), x > 0: y = \log_b x\}, \text{ for some } 0 < b \neq 1]$$

(d) the power functions with positive arguments

$$[f = \{(x, y), x > 0: y = x^u\}, \text{ for some } u]$$

These are very simple functions. All are continuous and, except for the exponential function with base 1 and the power function with exponent 0, all are monotonic. For each of the four kinds of functions, we have proved a number of useful theorems. For example [one theorem for each kind of function],

by <u>the ℓdpma</u>, if f is a homogeneous linear function then

(1) $$\forall_x \forall_y\ f(x + y) = f(x) + f(y),$$

by <u>Theorem 205</u>, if f is an exponential function then

(2) $$\forall_u \forall_v\ f(u + v) = f(u)f(v),$$

by <u>Theorem 215</u>, if f is a logarithm function then

(3) $$\forall_{u>0} \forall_{v>0}\ f(uv) = f(u) + f(v) \qquad [\text{and } \mathcal{D}_f = \{u: u > 0\}]$$

and by <u>Theorem 218</u>, if f is a power function with positive arguments then

(4) $$\forall_{u>0} \forall_{v>0}\ f(uv) = f(u)f(v) \qquad [\text{and } \mathcal{D}_f = \{u: u > 0\}].$$

It is interesting--and, as shown in section 9.10, proves to be useful--to discover that each function f which is, say, monotonic and which satisfies either (1), (2), (3), or (4) is, respectively, either a homogeneous linear function, an exponential function, a logarithm function, or a power function with positive arguments. As a matter of fact, we shall

prove much more than this. Here is a sample of what we shall prove:

> Each function f which satisfies (1) and is <u>not</u> a homogeneous linear function either is the constant function whose value is 0 or has the property that each square region of the number plane [whose sides are parallel to the axes], no matter how small it is and no matter where it is in the plane, contains an ordered pair belonging to f.

Functions which have the peculiar property described above must be quite complicated. You may even doubt that there can be functions of this kind, but, in a moment we will give an example of such a function. Before doing so, let's see that such a function cannot be monotonic. For this, it is sufficient to note that a function which contains ordered pairs in both the second and fourth quadrant cannot be increasing [Explain.], and that one which contains ordered pairs in both the first and third quadrants cannot be decreasing. So, given any monotonic function f, there is at least one whole quadrant which contains no ordered pair belonging to f. Hence, no monotonic function can contain ordered pairs in each square region of the number plane. [Since the constant function 0 is not monotonic, it follows from the theorem stated above that each monotonic function which satisfies (1) is a homogeneous linear function.]

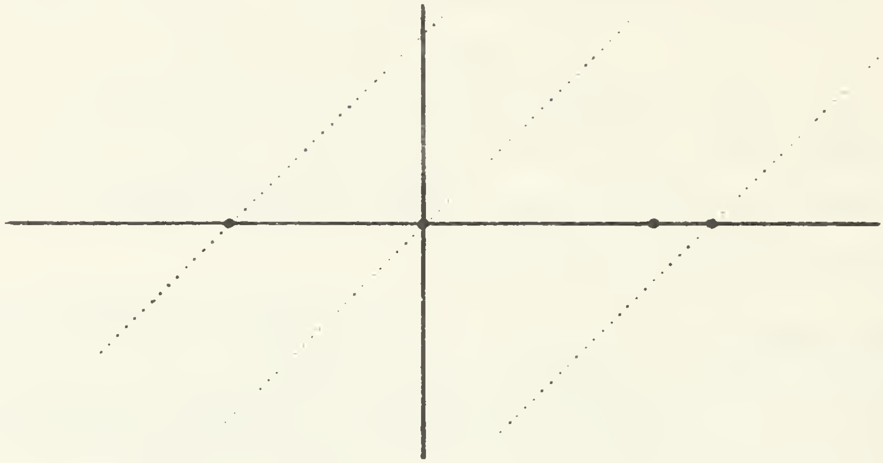
A QUEER FUNCTION

Although there actually are very many functions which <u>both</u> satisfy (1) <u>and</u> have the peculiar property we have been discussing, such functions are very difficult to define. Fortunately, it is not difficult to define a function which has the property in question and "partly satisfies (1)". [What this last means will appear when we have defined the function.] Here is a definition of such a function g:

> If there are rational numbers r and s such that $x = r + s\sqrt{2}$ then $g(x) = r$; if there are no such rational numbers r and s then $g(x) = 0$.

For example, for each rational number r, $g(r) = r$, $g(r - \sqrt{2}) = r$, and $g(r + \frac{147}{99}\sqrt{2}) = r$. Since there are no rational numbers r and s such that

$r + s\sqrt{2} = \sqrt{3}$, $g(\sqrt{3}) = 0$.[†] The figure is meant to suggest the ordered pairs of g just mentioned as examples. The lines are dotted to indicate



that the only points on them which belong to the graph of g are those at rational distances from the x-axis. You can imagine the complete graph of g to be constructed as follows:

> First, plot all the "rational points" on the graph of 'y = x', and call this set 'S'. Next, plot all the points on the x-axis whose abscissas are rational multiples of $\sqrt{2}$, and call this set 'T'. Next, plot all the points that can be obtained by shifting S to the left or right [but not up or down] so that the shifted set contains one of the points of T. [Do this for each point of T.]. Now, in your imagination, you have plotted all the ordered pairs (x, g(x)) such that, for some r and s, $x = r + s\sqrt{2}$. Complete the job by plotting the points (x, 0) for which there are no rational numbers r and s such that $x = r + s\sqrt{2}$. [For example, plot $(\sqrt{3}, 0)$.]

Of course, if you tried to carry out these instructions on paper, the resulting picture would be a mess, and you are probably wondering whether g really is a function. Is it really the case that no two ordered pairs of g have the same first component? To see that this is so, all we

---

[†]If there were rational numbers r and s such that $\sqrt{3} = r + s\sqrt{2}$, there would, then, be a rational number $s \neq 0$ such that $\sqrt{3} - s\sqrt{2}$ is a rational number. The square of this number--$3 - 2s\sqrt{6} + 2s^2$--would be rational and, hence, $2s\sqrt{6}$ would be rational. Since $s \neq 0$, it follows that $\sqrt{6}$ is rational. But, this is not the case.

need do is make sure that if x is a number for which there are rational numbers r_1 and s_1 such that $x = r_1 + s_1\sqrt{2}$ then there are not any other rational numbers--say r_2 and s_2--such that $x = r_2 + s_2\sqrt{2}$. This will show that there is no number x for which the definition tells us that $g(x) = r_1$ and $g(x) = r_2$ where $r_2 \neq r_1$. So, what we need do is show that

$$\text{if } r_2 + s_2\sqrt{2} = r_1 + s_1\sqrt{2} \text{ then } r_2 = r_1.$$

This is easy. For, suppose that $r_2 + s_2\sqrt{2} = r_1 + s_1\sqrt{2}$. Then, $(s_2 - s_1)\sqrt{2} = r_1 - r_2$ and, unless $s_2 - s_1 = 0$, it follows that $\sqrt{2} = (r_1 - r_2)/(s_2 - s_1)$, a rational number. So, since $\sqrt{2}$ is irrational, it follows that $s_2 = s_1$. Hence, since $r_2 + s_2\sqrt{2} = r_1 + s_1\sqrt{2}$, it follows that $r_2 = r_1$. [So, we could have obtained a queer function of the same kind as g by replacing '$\sqrt{2}$' in the definition of g by a name for any irrational number. The new function would be different from our function g if the irrational number we chose was not of the form '$r + s\sqrt{2}$'.]

The description of how g might be graphed probably convinced you that g does have the odd property we are interested in--that is, that each square region of the number plane contains an ordered pair belonging to g. At any rate, it is not difficult to prove that this is the case. To do so, consider, for any given numbers $d > 0$, x_0, and y_0, the square region whose center is (x_0, y_0) and whose side-measure is $2d$. [Draw a picture.] What we wish to do is find a number x such that $(x, g(x))$ belongs to this region--that is, such that

$$x_0 - d < x < x_0 + d \qquad \text{and} \qquad y_0 - d < g(x) < y_0 + d.$$

This is easy. By definition, if r is any rational number such that $y_0 - d < r < y_0 + d$, and s is any rational number at all, then, since $g(r + s\sqrt{2}) = r$,

$$y_0 - d < g(r + s\sqrt{2}) < y_0 + d.$$

By Theorem 201, there are rational numbers between $y_0 - d$ and $y_0 + d$. Let's choose r to be any one of them. All we have to do now is choose for s some rational number such that, for the chosen r,

$$x_0 - d < r + s\sqrt{2} < x_0 + d$$

--that is, such that

$$\frac{x_0 - d - r}{\sqrt{2}} < s < \frac{x_0 + d - r}{\sqrt{2}}$$

Again, by Theorem 201, whatever number r we have chosen, there are such numbers s, and we choose any one of them. So, with such choices of r and s, the pair $(r + s\sqrt{2},\ g(r + s\sqrt{2}\,))$ does belong to the given square region. Consequently, g does in fact have the queer property we wished it to have.

As remarked at the beginning, g does not quite satisfy (1) [although there are other functions which have the queer property g has and do satisfy (1)]. That g does not satisfy (1) can be shown by finding two numbers x and y such that $g(x + y) \neq g(x) + g(y)$. Now, by definition, $g(1 + \sqrt{2}\,) = 1$ and $g(\sqrt{3}\,) = 0$. But, since there are no rational numbers r and s such that $1 + \sqrt{2} + \sqrt{3} = r + s\sqrt{2}$, $g((1 + \sqrt{2}\,) + \sqrt{3}\,) = 0 \neq 1 + 0$. [If there were rational numbers r and s such that $1 + \sqrt{2} + \sqrt{3} = r + s\sqrt{2}$ then it would follow that $\sqrt{3} = (r - 1) + (s - 1)\sqrt{2}$. Since $r - 1$ and $s - 1$ would be rational numbers, this is impossible.]

However, there is a subset f of g which has all the queerness of g and is such that

$$(1') \qquad \forall_{x \in \mathfrak{D}_f} \forall_{y \in \mathfrak{D}_f} (x + y \in \mathfrak{D}_f \text{ and } f(x + y) = f(x) + f(y)).$$

One such function f is the one whose graph consists of only those points of the graph of g which are on the slanting lines. That is,

$$\mathfrak{D}_f = \{x\colon \exists_r \exists_s\, x = r + s\sqrt{2}\,\} \text{ and } \forall_r \forall_s\, f(r + s\sqrt{2}\,) = r.$$

That f is a function follows from the fact that f is a subset of g and g is a function. The proof that f is queer is the same as the proof, above, that g is queer [In fact, the ordered pair of g which we located in the square region with center (x_0, y_0) and side-measure 2d also belongs to f.].

So, all that remains is to show that f satisfies (1′). We begin by noting that, from the definition of $\mathfrak{D}_f$, it follows that, since the set R of rational numbers is closed with respect to addition, $\mathfrak{D}_f$ is also closed with respect to addition. So, f satisfies the first part of (1′). We end by noting that, if $x = r_1 + s_1\sqrt{2}$ and $y = r_2 + s_2\sqrt{2}$ then $x + y = (r_1 + r_2) + (s_1 + s_2)\sqrt{2}$ and, so, by definition, $f(x) = r_1$, $f(y) = r_2$, and $f(x + y) = r_1 + r_2$. Hence, for each x and y in $\mathfrak{D}_f$, $f(x + y) = f(x) + f(y)$--that is, f satisfies the second part of (1′).

A THEOREM ON HOMOGENEOUS LINEAR FUNCTIONS

Having considered some examples, it's now time to get back to our program of proving some theorems like the one displayed on page 9-315. In applications of mathematics one often has to deal with functions which are not defined for all real numbers and which, just for this reason, cannot possibly satisfy:

$$(1) \qquad \forall_x \forall_y \ f(x + y) = f(x) + f(y)$$

However, many of them [like the particular function f of our example] satisfy:

$$(1') \qquad \forall_{x \in \mathfrak{D}_f} \forall_{y \in \mathfrak{D}_f} (x + y \in \mathfrak{D}_f \text{ and } f(x + y) = f(x) + f(y))$$

To be more explicit, some problems in, for example, physics or chemistry, can be solved by discovering, experimentally, that one quantity is a function of another and that the function in question satisfies (1′). If this is the case then, using the theorem we are going to prove, it is possible to conclude that the function in question is a homogeneous linear function restricted to $\mathfrak{D}_f$, and this furnishes a way of predicting the value of the first of the two quantities when one knows the corresponding value of the second. In brief, one has discovered a law of nature. An example of this sort of discovery is given in the discussion of Gay-Lussac's Law which begins on page 9-139.

The functions which arise in applications of the kind described above, besides satisfying (1′), have the further property of being defined for at least all the points of some interval. Our theorem will tell us that each such function is either a homogeneous linear function restricted to $\mathfrak{D}_f$, or the constant function whose value is 0 and whose domain is $\mathfrak{D}_f$, or a function which, for arguments in as short an interval as you wish, has values as close as you wish to each real number. This suffices for the applications, since physical [or chemical] considerations will, in each case, show that the experimentally determined function can neither be constant nor extremely "jumpy". [In fact, in most cases it will be clear that it is monotonic.]

We are now ready for the statement of our basic theorem on homogeneous linear functions. [From it, we will derive similar theorems about exponential, logarithmic, and power functions.]

<u>Theorem 221.</u>

If f is a function such that

(1') $\forall_{x \in \mathfrak{D}_f} \forall_{y \in \mathfrak{D}_f} (x + y \in \mathfrak{D}_f \text{ and } f(x+y) = f(x) + f(y))$

then either f is a subset of a homogeneous linear function or f is not monotonic. In the latter case, either f is a subset of the constant function 0 or, for each x_0 which belongs to an interval contained in $\mathfrak{D}_f$ and for each y_0 and each $d > 0$, there is an $x \in \mathfrak{D}_f$ such that

$$x_0 - d < x < x_0 + d \text{ and } y_0 - d < f(x) < y_0 + d.$$

[Note that replacing (1'), above, by:

(1) $$\forall_x \forall_y\ f(x + y) = f(x) + f(y),$$

amounts to adding the hypothesis that $\mathfrak{D}_f$ is the set of all real numbers. So, the theorem which results when (1') is replaced by (1) is a corollary of Theorem 221. In this case, since $\mathfrak{D}_f$ is the set of all real numbers, each number x_0 automatically belongs to an interval contained in $\mathfrak{D}_f$. So, the corollary is just a restatement of the theorem displayed on page 9-315. Note, also, that the second sentence in Theorem 221 tells you nothing unless $\mathfrak{D}_f$ contains an interval. But, whatever $\mathfrak{D}_f$ is, the first sentence tells you that if f is monotonic and satisfies (1') then f is a subset of a homogeneous linear function.]

In order to simplify the proof of Theorem 221, we begin by proving three lemmas concerning any function f which satisfies (1').

<u>Lemma 1</u>. $0 \in \mathfrak{D}_f \Rightarrow f(0) = 0$

<u>Lemma 2</u>. $\forall_{x \in \mathfrak{D}_f} \forall_n (nx \in \mathfrak{D}_f \text{ and } f(nx) = nf(x))$

<u>Lemma 3</u>. For each x_1 and x_2 in $\mathfrak{D}_f$ and each r_1 and r_2 such that $r_1x_1 + r_2x_2 \in \mathfrak{D}_f$, $f(r_1x_1 + r_2x_2) = r_1f(x_1) + r_2f(x_2)$

[Only Lemmas 1 and 2 are needed for the proof of the first part of Theorem 221. Go over the proofs, given below, for Lemmas 1 and 2 now, and make sure you understand Lemma 3. Then you may find it simpler to skip to the proof of Theorem 221 and, afterwards, study the proof of Lemma 3. The latter is somewhat tedious.]

Proof of Lemma 1. Since f satisfies (1′), it follows that, if $0 \in \mathcal{D}_f$ then

$$f(0) = f(0 + 0) = f(0) + f(0).$$

So, if $0 \in \mathcal{D}_f$ then $f(0) = 0$.

Proof of Lemma 2. [By induction.] For $a \in \mathcal{D}_f$, since $1 \cdot a = a$, it follows that $1 \cdot a \in \mathcal{D}_f$ and $f(1 \cdot a) = f(a) = 1 \cdot f(a)$.

For $a \in \mathcal{D}_f$, suppose that $pa \in \mathcal{D}_f$ and $f(pa) = pf(a)$. Since pa and a belong to $\mathcal{D}_f$, it follows from (1′) that

$$pa + a \in \mathcal{D}_f \quad \text{and} \quad f(pa + a) = f(pa) + f(a).$$

Consequently, $(p + 1)a \in \mathcal{D}_f$ and, since $f(pa) = pf(a)$,

$$f((p+1)a) = f(pa+a) = f(pa) + f(a) = pf(a) + f(a) = (p+1)f(a).$$

So, by mathematical induction, Lemma 2.

Proof of Lemma 3. Consider arguments a_1 and a_2 of f and rational numbers r_1 and r_2 such that $r_1a_1 + r_2a_2 \in \mathcal{D}_f$. Choose a positive integer p such that $p + r_1 > 0$ and $p + r_2 > 0$ and choose a positive integer m such that $r_1m \in I$ and $r_2m \in I$.

By Lemma 2, since $a_1 \in \mathcal{D}_f$ and $a_2 \in \mathcal{D}_f$, it follows that $pa_1 \in \mathcal{D}_f$ and $pa_2 \in \mathcal{D}_f$, and, so, by (1′), that $pa_1 + pa_2 \in \mathcal{D}_f$. Hence, since $r_1a_1 + r_2a_2 \in \mathcal{D}_f$, it follows, by (1′), that $(pa_1 + pa_2) + (r_1a_1 + r_2a_2)$ belongs to $\mathcal{D}_f$.

This being the case, it follows, by (1′), that

$$\begin{aligned} f((pa_1+pa_2) + (r_1a_1+r_2a_2)) &= f(pa_1+pa_2) + f(r_1a_1+r_2a_2) \\ &= f(pa_1) + f(pa_2) + f(r_1a_1+r_2a_2). \end{aligned}$$

Since $(pa_1 + pa_2) + (r_1a_1 + r_2a_2) = (p + r_1)a_1 + (p + r_2)a_2$ and since, by Lemma 2, $f(pa_1) = pf(a_1)$ and $f(pa_2) = pf(a_2)$, it follows that

(i) $\quad f((p+r_1)a_1 + (p+r_2)a_2) = pf(a_1) + pf(a_2) + f(r_1a_1+r_2a_2).$

Also, by Lemma 2, since $(p + r_1)a_1 + (p + r_2)a_2 \in \mathcal{D}_f$, it follows that

$$f(m[(p+r_1)a_1 + (p+r_2)a_2]) = mf((p+r_1)a_1 + (p+r_2)a_2).$$

Since $m[(p + r_1)a_1 + (p + r_2)a_2] = m(p + r_1)a_1 + m(p + r_2)a_2$,

it follows that

(ii) $f(m(p+r_1)a_1 + m(p+r_2)a_2) = mf((p+r_1)a_1 + (p+r_2)a_2)$.

Now, since $r_1 m \in I$, it follows that $m(p + r_1) \in I$ and, since $p + r_1 > 0$, it follows that $m(p + r_1) \in I^+$. So, by Lemma 2, since $a_1 \in \mathcal{D}_f$, it follows that $m(p + r_1)a_1 \in \mathcal{D}_f$. Similarly, $m(p + r_2) \in I^+$ and $m(p + r_2)a_2 \in \mathcal{D}_f$. Hence, by (1′),

$$f(m(p+r_1)a_1 + m(p+r_2)a_2) = f(m(p+r_1)a_1) + f(m(p+r_2)a_2).$$

So, by Lemma 2 [since a_1 and a_2 are arguments of f and $m(p + r_1)$ and $m(p + r_2)$ are positive integers], it follows that

(iii) $f(m(p+r_1)a_1 + m(p+r_2)a_2) = m(p+r_1)f(a_1) + m(p+r_2)f(a_2)$.

Comparing (ii) and (iii), and noting that $m \neq 0$, we see that

$$\begin{aligned} f((p+r_1)a_1 + (p+r_2)a_2) &= (p+r_1)f(a_1) + (p+r_2)f(a_2) \\ &= pf(a_1) + pf(a_2) + r_1 f(a_1) + r_2 f(a_2). \end{aligned}$$

Comparing this last with (i), we see that

$$f(r_1 a_1 + r_2 a_2) = r_1 f(a_1) + r_2 f(a_2).$$

Proof of Theorem 221.

Suppose that f satisfies (1′) but is not a subset of a homogeneous linear function. For the first part of the theorem, we wish to show that f is not monotonic. For the second part, we shall assume, also, that f is not a subset of the constant 0, and show that, for each x_0 which belongs to an interval contained in $\mathcal{D}_f$ and for each y_0 and each $d > 0$, there is an $x \in \mathcal{D}_f$ such that $|x - x_0| < d$ and $|f(x) - y_0| < d$.

Since f is not a subset of a homogeneous linear function, it follows that f has nonzero arguments. For, if $\mathcal{D}_f = \emptyset$ then, trivially, for each $x \in \mathcal{D}_f$, $f(x) = x$; and, if $\mathcal{D}_f = \{0\}$ then, by Lemma 1, for each $x \in \mathcal{D}_f$, $f(x) = x$--in either case, f is a subset of a homogeneous linear function. Since, as we have shown, f has a nonzero argument and since $\mathcal{D}_f$ is closed with respect to addition, it follows that f has many arguments. If, for all of its arguments, f has the value 0 then [since f has at least two arguments] f is not monotonic. Since this is the conclusion we wish to reach in the first part of the proof, we may, for this part, assume that f has an argument x_1 such that $f(x_1) \neq 0$. And, for the second part

of the proof, this follows from our assumption that f is not a subset of the constant 0. So, from now on, we shall assume that x_1 is an argument of f such that $f(x_1) \neq 0$.

Setting $c = f(x_1)/x_1$, it follows that $c \neq 0$ and $f(x_1) = cx_1$. Since, by hypothesis, f is not a subset of a homogeneous linear function, it follows that f has an argument x_2 such that $f(x_2) \neq cx_2$. By Lemma 1, $x_2 \neq 0$. Since $f(x_1) = cx_1$, $x_2f(x_1) = cx_1x_2$. Since $f(x_2) \neq cx_2$ and $x_1 \neq 0$, $x_1f(x_2) \neq cx_1x_2$. So, there are nonzero arguments x_1 and x_2 of f such that $f(x_1) \neq 0$ and

$$\text{(i)} \qquad x_1f(x_2) \neq x_2f(x_1).$$

We shall now use (i) to show that f is not monotonic, thus completing the proof of the first part of Theorem 221. After doing so, we shall use (i), in another way, to prove the second part of Theorem 221.

Since f satisfies (1′), it follows that $-f$ satisfies (1′). Also, if $-f$ is not monotonic then neither is f. So, without loss of generality, we may assume that $f(x_1) > 0$. [If $f(x_1) < 0$, we merely transfer our attention to $-f$.] Also, we may, without loss of generality, assume that x_1 and x_2 are both positive. For, to begin with, suppose that $x_2 < 0 < x_1$. In this case there is a positive integer p such that $x_2 + px_1 > 0$. By Lemma 2 and (1′), it follows that $f(x_2 + px_1) = f(x_2) + pf(x_1)$. So,

$$x_1f(x_2 + px_1) = x_1f(x_2) + px_1f(x_1).$$

Since $$(x_2 + px_1)f(x_1) = x_2f(x_1) + px_1f(x_1),$$

it follows from (i) that

$$x_1f(x_2 + px_1) \neq (x_2 + px_1)f(x_1).$$

So, if $x_2 < 0 < x_1$, we can replace x_2 by $x_2 + px_1$ and have two positive arguments of f which satisfy (i). Note that one of these arguments is x_1 and, as before, $f(x_1) \neq 0$. Next, suppose that $x_1 < 0 < x_2$. In this case we can choose p so that $x_2 + px_1 < 0$ and end up with two negative arguments of f, one of which is x_1, which satisfy (i). So, without loss of generality, we may assume that x_1 and x_2 are either both positive or both negative. Now, since f satisfies (1′) so does the function g where $g(x) = f(-x)$. And, if g is not monotonic then neither is f. So, without loss of generality, we may assume that both x_1 and x_2 are positive. [If they are both negative, we merely transfer our attention to g.]

There are now, two cases to consider:

$$\frac{x_2}{x_1} < \frac{f(x_2)}{f(x_1)} \quad \text{and:} \quad \frac{x_2}{x_1} > \frac{f(x_2)}{f(x_1)}$$

Since $x_2/x_1 > 0$, there are, in either case, positive integers m and n such that m/n is between x_2/x_1 and $f(x_2)/f(x_1)$. In the first case

(a) $$\frac{x_2}{x_1} < \frac{m}{n} < \frac{f(x_2)}{f(x_1)}.$$

In this case it follows [since $x_1 > 0$ and $n > 0$] that $nx_2 < mx_1$ and [since $n > 0$ and $f(x_1) > 0$] that $nf(x_2) > mf(x_1)$. From the last, by Lemma 2, $f(nx_2) > f(mx_1)$. So, f is not an increasing function. To show that f is not decreasing, let p be a positive integer such that $mx_1 < px_2$. Since $f(x_1) > 0$ and $f(x_2)/f(x_1) > m/n > 0$, it follows that $f(x_2) > 0$. And, since $nx_2 < mx_1 < px_2$ and $x_2 > 0$, it follows that $n < p$. So, by Lemma 2,

$$f(mx_1) = mf(x_1) < nf(x_2) < pf(x_2) = f(px_2).$$

Since $mx_1 < px_2$ and $f(mx_1) < f(px_2)$, it follows that f is not decreasing. Since f is neither increasing nor decreasing, it follows that [in case (a)] f is not monotonic.

The alternative to (a) is that

(b) $$\frac{f(x_2)}{f(x_1)} < \frac{m}{n} < \frac{x_2}{x_1}.$$

It follows, as before, that $mx_1 < nx_2$ and that $f(mx_1) > f(nx_2)$. So, f is certainly not an increasing function. To show that f is not decreasing, choose p so that $nx_2 < px_1$. Then, since $m < p$ and $f(x_1) > 0$,

$$f(nx_2) < f(mx_1) = mf(x_1) < pf(x_1) = f(px_1).$$

Since $nx_2 < px_1$ and $f(nx_2) < f(px_1)$, f is not decreasing. Hence, in case (b), f is not monotonic.

This completes the proof of the first part of Theorem 221. To prove the second part, we assume that f satisfies (1′), is not a subset of a homogeneous linear function, and, also, is not a subset of the constant 0. As previously shown, it follows that there are arguments x_1 and x_2 of f such that

(i) $$x_1 f(x_2) \neq x_2 f(x_1).$$

Suppose, now, that x_0 is any number which belongs to an interval

all of whose members are arguments of f and that y_0 is any number at all. Suppose, also, that d is any positive number. We shall show that there are rational numbers r_1 and r_2 such that if $x = r_1x_1 + r_2x_2$ then $x \in \mathcal{D}_f$ and

$$x_0 - d < x < x_0 + d \quad \text{and} \quad y_0 - d < f(x) < y_0 + d.$$

Notice that, since x_0 belongs to an interval all of whose members are arguments of f, the first of the pairs of inequations just displayed will ensure that $x \in \mathcal{D}_f$ if d is a sufficiently small positive number. Without loss of generality, we can suppose that d is small enough for this to be the case. Notice, also, that, since x_1 and x_2 are arguments of f and since r_1 and r_2 are to be chosen so that $r_1x_1 + r_2x_2 \in \mathcal{D}_f$, it follows from Lemma 3 that f(x)--that is $f(r_1x_1 + r_2x_2)$--will be $r_1f(x_1) + r_2f(x_2)$. So, our job is just, given x_0, y_0, and $d > 0$, to find r_1 and r_2 such that

(ii) $x_0 - d < r_1x_1 + r_2x_2 < x_0 + d$ and $y_0 - d < r_1f(x_1) + r_2f(x_2) < y_0 + d$.

We note first that, because of (i), the simultaneous equations in (a, b):

$$\text{(iii)} \qquad \begin{cases} x_1a + x_2b = x_0 \\ f(x_1)a + f(x_2)b = y_0 \end{cases}$$

have a unique solution. If the components of this solution happened to be rational numbers r_1 and r_2 then things would be very simple--for then the value of f at x_0, itself, would be precisely y_0 and (ii) would be satisfied. This is not likely to be the case, but (ii) leaves us enough leeway to find a pair (r_1, r_2) of rational numbers which comes near enough to satisfying (iii) that it will also satisfy (ii). Briefly, if the solution of (iii) is (a, b) and r_1 and r_2 are chosen sufficiently near a and b, respectively, then $x_1r_1 + x_2r_2$ will be as close as we like to x_0 and $f(x_1)r_1 + f(x_2)r_2$ will be as close as we wish to y_0.

More explicitly, for any rational numbers r_1 and r_2, by (iii),

$$x_1r_1 + x_2r_2 = x_0 + x_1(r_1 - a) + x_2(r_2 - b)$$

and

$$f(x_1)r_1 + f(x_2)r_2 = y_0 + f(x_1)(r_1 - a) + f(x_2)(r_2 - b).$$

From this one sees that if $|r_1 - a|$ and $|r_2 - b|$ are sufficiently small [how small they need be depends on x_1, x_2, $f(x_1)$ and $f(x_2)$] then (ii) will be satisfied.

This completes the proof of Theorem 221.

THREE OTHER THEOREMS

Using Theorem 221 and some theorems concerning logarithmic and exponential functions it is not difficult to derive theorems about functions which satisfy:

$$(2) \qquad \forall_u \forall_v \; f(u + v) = f(u)\, f(v)$$

or, the less restrictive:

$$(2') \qquad \forall_{u \in \mathfrak{D}_f} \forall_{v \in \mathfrak{D}_f} \, (u + v \in \mathfrak{D}_f \text{ and } f(u + v) = f(u)f(v))$$

As indicated earlier, one might expect the solutions of (2) to be exponential functions. The discussion leading up to Theorem 221, however, should prepare us to expect that (2) also has some queer solutions. This is the case.

In Theorem 222 [see below] we assume, besides that f satisfies (2′), that each value of f is positive. As shown, following the proof of Theorem 222, this latter assumption is unnecessary in case f satisfies (2)--that is, in case f satisfies (2′) and $\mathfrak{D}_f$ is the set of all real numbers.

<u>Theorem 222.</u>

If f is a function whose values are positive and is such that

$$(2') \quad \forall_{u \in \mathfrak{D}_f} \forall_{v \in \mathfrak{D}_f} \, (u + v \in \mathfrak{D}_f \text{ and } f(u + v) = f(u)f(v))$$

then either f is a subset of an exponential function with positive base different from 1 or f is not monotonic. In the latter case either f is a subset of the constant function 1 or, for each u_o which belongs to an interval contained in $\mathfrak{D}_f$ and for each $v_o > 0$ and each $d > 0$, there is a $u \in \mathfrak{D}_f$ such that

$$u_o - d < u < u_o + d \text{ and } v_o - d < f(u) < v_o + d.$$

<u>Proof of Theorem 222.</u>

Since all values of f are positive, the function F for which

$$F(u) = \ell n\,(f(u))$$

has the same domain as f. So, for $u \in \mathfrak{D}_F$ and $v \in \mathfrak{D}_F$, $u + v \in \mathfrak{D}_F$ and

$$\left.\begin{aligned} F(u+v) &= \ln[f(u+v)] \\ &= \ln[f(u)f(v)] \end{aligned}\right\} (2')$$

$$\begin{aligned} &= \ln(f(u)) + \ln(f(v)) \\ &= F(u) + F(v). \end{aligned}$$

Consequently, F satisfies (1′). Note, also, that since $\ln$ is increasing, F is monotonic if and only if f is monotonic.

It follows from these remarks and the first part of Theorem 221 that if f is monotonic then F is a subset of a homogeneous linear function--that is, there is a $c \neq 0$ such that

$$\forall_{u \in \mathcal{D}_F} \ln(f(u)) = cu.$$

Since $\mathcal{D}_F = \mathcal{D}_f$, it follows that, for each $u \in \mathcal{D}_f$,

$$f(u) = e^{cu} = a^u, \quad \text{with } a = e^c.$$

Since $c \neq 0$, $0 < a \neq 1$. This completes the proof of the first part of Theorem 222.

To prove the second part we use the second part of Theorem 221. This, together with the fact that $\mathcal{D}_F = \mathcal{D}_f$ and that F is monotonic if and only if f is, tells us that if f is not monotonic then either F is a subset of the constant function 0--in which case f is a subset of the constant function 1--or, for each u_o which belongs to an interval contained in $\mathcal{D}_f$ and for each $v_o > 0$ and each $d' > 0$, there is a $u \in \mathcal{D}_f$ such that

$$u_o - d' < u < u_o + d' \text{ and } \ln v_o - d' < \ln(f(u)) < \ln v_o + d'.$$

Since the exponential function is continuous at $\ln v_o$, given any $d > 0$, there is a $d' > 0$ such that

$$(*) \qquad \text{if } |\ln(f(u) - \ln v_o| < d' \text{ then } |f(u) - v_o| < d.$$

So, if, given $d > 0$, we choose for the d' of the second case a number which is less than or equal to d and satisfies (*) then the second case tells us that there is a $u \in \mathcal{D}_f$ such that

$$u_o - d < u < u_o + d \text{ and } v_o - d < f(u) < v_o + d.$$

This completes the proof of Theorem 222.

It should be remarked that if one assumes more about $\mathcal{D}_f$ one can, in view of (2′), dispense with the assumption that the values of f are

positive. Suppose, for example, that f is defined for all real numbers and satisfies (2′)--that is, that f satisfies (2) on page 9-314. In this case, for each u, $u/2 \in \mathfrak{D}_f$ and, so, by (2′),

$$f(u) = f\left(\frac{u}{2} + \frac{u}{2}\right) = \left[f\left(\frac{u}{2}\right)\right]^2 \geq 0.$$

So, in this case, (2′) [or (2)] ensures that the values of f are at least nonnegative. Now, suppose that for some v_0, $f(v_0) = 0$. Since, for each u, $u - v_0 \in \mathfrak{D}_f$, it follows, again by (2′), that, for each u,

$$f(u) = f((u - v_0) + v_0) = f(u - v_0)f(v_0) = 0.$$

So, if f satisfies

(2) $\qquad \forall_u \forall_v\, f(u + v) = f(u)f(v)$

then either f is the constant function 0 or all values of f are positive. Since a function which satisfies (2) must also satisfy (2′), Theorem 222 tells us what happens in the second case--f is either an exponential function with positive base, or a queer function.

A similar remark applies to situations in which it is known that $\mathfrak{D}_f$ is the set of nonnegative numbers and that

(**) $\qquad \forall_{u \geq 0} \forall_{v \geq 0}\, f(u + v) = f(u)f(v).$

As before, this implies that the values of f are all nonnegative. Also, as before, if $f(v_0) = 0$ then, for each $u > v_0$, $f(u) = 0$. Now, suppose that $f(v_0) = 0$ and that there is a $u_0 > 0$ such that $f(u_0) \neq 0$. [By the results just noted, $u_0 < v_0$, and $f(u_0) > 0$.] It is easy to prove, using (**), that, for each n, $f(nu_0) = [f(u_0)]^n$. [See the proof of Lemma 2 on page 9-321.] So, since $f(u_0) \neq 0$, it follows that, for each n, $f(nu_0) \neq 0$. However, for $n = [\![v_0/u_0]\!] + 1$, $nu_0 > v_0$ and, so, for this n, $f(nu_0) = 0$. Consequently, (**) implies that either, for each $u > 0$, $f(u) = 0$ or, for each $u > 0$, $f(u) > 0$. Also, it follows from (**) that $f(0) = f(0 + 0) = [f(0)]^2$ and, so, that either $f(0) = 0$ or $f(0) = 1$. As we have seen [$v_0 = 0$], if $f(0) = 0$ then $f(u) = 0$ for each $u \geq 0$--that is, f is the constant function whose domain is the set of nonnegative numbers and whose value is 0. On the other hand, if $f(0) = 1$ then, as we have seen, either, for each $u > 0$, $f(u) = 0$--that is, f is the exponential function with base 0--or, for each $u > 0$, $f(u) > 0$. In this last case, since, also, $f(0) = 1 > 0$, Theorem 222 tells us that f is either an exponential function with positive base restricted to nonnegative arguments or is a queer function.

Summarizing, if $\mathfrak{D}_f$ is the set of nonnegative numbers and f satisfies (**) then, either f is an exponential function with nonnegative base restricted to nonnegative arguments, or is the constant 0 with nonnegative arguments, or is a queer function.

Finally, a portion of the preceding argument suffices to show that if $\mathfrak{D}_f$ is the set of positive numbers and f satisfies (2′) then f is one of the functions just mentioned [but restricted to positive arguments].

In a very similar manner, we can derive a theorem about functions which satisfy a generalization of:

$$(3) \qquad \forall_u \forall_v \, f(uv) = f(u) + f(v).$$

As we expect, the "nice" solutions of (3) are logarithm functions.

<u>Theorem 223.</u>

If f is a function whose arguments are positive and is such that

$$(3') \quad \forall_{u \in \mathfrak{D}_f} \forall_{v \in \mathfrak{D}_f} (uv \in \mathfrak{D}_f \text{ and } f(uv) = f(u) + f(v))$$

then either f is a subset of a logarithm function [to a positive base different from 1] or f is not monotonic. In the latter case either f is a subset of the constant function 0 or, for each u_o which belongs to an interval contained in $\mathfrak{D}_f$ and for each v_o and each $d > 0$, there is a $u \in \mathfrak{D}_f$ such that

$$u_o - d < u < u_o + d \text{ and } v_o - d < f(u) < v_o + d.$$

<u>Proof of Theorem 223.</u>

Since all arguments of f are positive and since, for each $u > 0$, $u = e^{\ln u}$, the function F for which

$$F(x) = f(e^x)$$

has for its domain $\{x: \exists_{u \in \mathfrak{D}_f} x = \ln u\}$. [Note that, for each $u \in \mathfrak{D}_f$, $f(u) = F(\ln u)$.] So, for any arguments x and y of F there are arguments u

and v of f such that $x = \ln u$ and $y = \ln v$. It follows that $x + y = \ln u + \ln v = \ln(uv)$ and since, by (3′), $uv \in \mathfrak{D}_f$, it follows that $x + y \in \mathfrak{D}_F$. Moreover,

$$\begin{aligned} F(x+y) &= f(e^{x+y}) \\ &= \left.\begin{aligned} &f(e^x \cdot e^y) \\ &= f(e^x) + f(e^y) \end{aligned}\right\} \quad (3') \\ &= F(x) + F(y). \end{aligned}$$

Consequently, F satisfies (1′). Note, also, that since the exponential function is increasing, F is monotonic if and only if f is monotonic.

If follows from these remarks and the first part of Theorem 221 that if f is monotonic then F is a subset of a homogeneous linear function--that is, there is a $c \neq 0$ such that

$$\forall_{u \in \mathfrak{D}_f}\ f(e^{\ln u}) = c \ln u.$$

Since $c \neq 0$, it follows that, for each $u \in \mathfrak{D}_f$,

$$f(u) = \log_b u, \quad \text{with } b = e^{1/c}.$$

This completes the proof of the first part of Theorem 223.

To prove the second part, we use the second part of Theorem 221. In view of the fact that F is monotonic if and only if f is, this tells us that if f is not monotonic then either F is a subset of the constant function 0--in which case, the same holds for f--or, for each x_0 which belongs to an interval contained in $\mathfrak{D}_F$ and for each v_0 and each $d' > 0$, there is an $x \in \mathfrak{D}_F$ such that

$$x_0 - d' < x < x_0 + d' \quad \text{and} \quad v_0 - d' < F(x) < v_0 + d'.$$

To see what this means for f, we begin by noting that, since $\ln$ is an increasing function, if a number u_0 belongs to an interval contained in $\mathfrak{D}_f$ then $\ln u_0$ belongs to an interval contained in $\mathfrak{D}_F$. So, taking $x_0 = \ln u_0$ and $x = \ln u$, we see, in the second case, that, for each u_0 which belongs to an interval contained in $\mathfrak{D}_f$ and for each v_0 and each $d' > 0$, there is a $u \in \mathfrak{D}_f$ such that

$$\ln u_0 - d' < \ln u < \ln u_0 + d' \quad \text{and} \quad v_0 - d' < f(u) < v_0 + d'.$$

Now, since the exponential function is continuous at $\ln u_0$, given any

$d > 0$, there is a $d' > 0$ such that

$$(*) \qquad \text{if } |\ln u - \ln u_0| < d' \text{ then } |u - u_0| < d.$$

So, if, given d, we choose for d' a number which is less than or equal to d and satisfies (*) then the second case tells us that there is a $u \in \mathfrak{D}_f$ such that

$$u_0 - d < u < u_0 + d \text{ and } v_0 - d < f(u) < v_0 + d.$$

This completes the proof of Theorem 223.

Next, we prove a theorem about power functions.

Theorem 224.

If f is a function whose arguments and values are positive and is such that

$$(4') \quad \forall_{u \in \mathfrak{D}_f} \forall_{v \in \mathfrak{D}_f} (uv \in \mathfrak{D}_f \text{ and } f(uv) = f(u)f(v))$$

then either f is a subset of a power function with non-zero exponent or f is not monotonic. In the latter case, either f is a subset of the constant function 1 or, for each u_0 which belongs to an interval contained in $\mathfrak{D}_f$ and for each $v_0 > 0$ and each $d > 0$, there is a $u \in \mathfrak{D}_f$ such that

$$u_0 - d < u < u_0 + d \text{ and } v_0 - d < f(u) < v_0 + d.$$

Proof of Theorem 224.

Since both the arguments and the values of f are positive, it follows, as in the proofs of Theorems 222 and 223, that the function F for which

$$F(x) = \ln(f(e^x))$$

has for its domain $\{x: \exists_{u \in \mathfrak{D}_f}\, x = \ln u\}$. [Note that, for each $u \in \mathfrak{D}_f$, $\ln(f(u)) = F(\ln u)$.] So, as in the proof of Theorem 223, if x and y are arguments of F, so is $x + y$. Moreover,

$$\left.\begin{aligned} F(x+y) &= \ln(f(e^{x+y})) \\ &= \ln(f(e^x \cdot e^y)) \\ &= \ln(f(e^x)\cdot f(e^y)) \end{aligned}\right\} (4') \\ \begin{aligned} &= \ln(f(e^x)) + \ln(f(e^y)) \\ &= F(x) + F(y). \end{aligned}$$

Consequently, F satisfies (1′). Note, also, that since both the exponential function and $\ln$ are increasing, F is monotonic if and only if f is monotonic.

It follows from these remarks and the first part of Theorem 221 that if f is monotonic then F is a subset of a homogeneous linear function--that is, that there is a $c \neq 0$ such that

$$\forall_{u \in \mathfrak{D}_f} \ln(f(e^{\ln u})) = c \ln u = \ln(u^c),$$

whence, for each $u \in \mathfrak{D}_f$, $f(u) = u^c$. This completes the proof of the first part of Theorem 224.

To prove the second part we use the second part of Theorem 221. In view of the fact that F is monotonic if and only if f is, this tells us that either F is a subset of the constant 0--in which case f is a subset of the constant function 1--or, for each x_o which belongs to an interval contained in $\mathfrak{D}_F$ and for each y_o and each $d' > 0$, there is an $x \in \mathfrak{D}_F$ such that

$$x_o - d' < x < x_o + d' \quad \text{and} \quad y_o - d' < F(x) < y_o + d'.$$

As in the proof of Theorem 223, if u_o belongs to an interval contained in $\mathfrak{D}_f$ then $\ln u_o$ belongs to an interval contained in $\mathfrak{D}_F$. So, if f is not a power function, we know that, for each u_o which belongs to an interval contained in $\mathfrak{D}_f$ and for each $v_o > 0$ and each $d' > 0$, there is a $u \in \mathfrak{D}_f$ such that

$$\ln u_o - d' < \ln u < \ln u_o + d' \quad \text{and} \quad \ln v_o - d' < \ln(f(u)) < \ln v_o - d'.$$

Since the exponential function is continuous at $\ln u_o$ and at $\ln v_o$, given $d > 0$, there is a $d_1 > 0$ and a $d_2 > 0$ such that

$$\text{if} \quad |\ln u - \ln u_o| < d_1 \quad \text{then} \quad |u - u_o| < d$$

and

$$\text{if} \quad |\ln(f(u)) - \ln v_o| < d_2 \quad \text{then} \quad |f(u) - v_o| < d.$$

So, if, given $d > 0$, we take for d' the smaller of d_1 and d_2, we know that

$$u_0 - d < u < u_0 + d \quad \text{and} \quad v_0 - d < f(u) < v_0 + d.$$

This completes the proof of Theorem 224.

As in the case of Theorem 222, if the domain of f is the set of positive numbers and f satisfies:

$$(4) \qquad \forall_{u>0} \forall_{v>0}\ f(uv) = f(u)f(v),$$

one can dispense with the assumption that the values of f are positive. In this case, for each $u > 0$, $\sqrt{u} \in \mathfrak{D}_f$ and, by (4),

$$f(u) = f(\sqrt{u}\sqrt{u}) = [f(\sqrt{u})]^2 \geq 0.$$

So, at any rate, the values of f are nonnegative. Also, if, for some $v_0 > 0$, $f(v_0) = 0$, it follows from (4) that, for all $u > 0$--that is, for all $u \in \mathfrak{D}_f$--

$$f(u) = f\left(\frac{u}{v_0} \cdot v_0\right) = f\left(\frac{u}{v_0}\right) f(v_0) = 0.$$

So, if f satisfies (4) then either f is the constant 0 whose domain is the set of positive numbers or each value of f is positive.

AN APPLICATION OF THE FOUR THEOREMS

One normally compares two numbers by considering either their difference or, in the case of positive numbers, their ratio. So, for a function g, one's standards of comparison for two values $g(x)$ and $g(y)$ are $g(x) - g(y)$ and, if $\mathfrak{R}_g$ consists of positive numbers, $g(x)/g(y)$. Often when such a function g crops up in applications, it turns out that one of the quantities $g(x) - g(y)$ or $g(x)/g(y)$ depends only on $x - y$ or, if $\mathfrak{D}_g$ consists of positive numbers, only on x/y. That is, often there will be a function f such that, either

$(*_1)$ $\forall_{x \in \mathfrak{D}_g} \forall_{y \in \mathfrak{D}_g}$ $(x - y \in \mathfrak{D}_f$ and $g(x) - g(y) = f(x - y))$, or

$(*_2)$ $\forall_{x \in \mathfrak{D}_g} \forall_{y \in \mathfrak{D}_g}$ $(g(y) > 0$ and $x - y \in \mathfrak{D}_f$ and $g(x)/g(y) = f(x - y))$, or

$(*_3)$ $\forall_{x \in \mathfrak{D}_g} \forall_{y \in \mathfrak{D}_g}$ $(y > 0$ and $x/y \in \mathfrak{D}_f$ and $g(x) - g(y) = f(x/y))$, or

$(*_4)$ $\forall_{x \in \mathfrak{D}_g} \forall_{y \in \mathfrak{D}_g}$ $(y > 0$ and $g(y) > 0$ and $x/y \in \mathfrak{D}_f$ and $g(x)/g(y) = f(x/y))$.

The functions g which are encountered in such applications are usually monotonic. In cases $(*_1)$ and $(*_2)$ their domains are closed with respect to addition and in cases $(*_3)$ and $(*_4)$ their domains are closed with respect to multiplication. If these supplementary conditions are met then, as is readily shown [see below], the function f is monotonic and satisfies (1′), (2′), (3′), or (4′), respectively. Consequently, in the four cases, respectively, the function g is a subset of

$(\dagger_1)$ a linear function, or of

$(\dagger_2)$ the product of a positive constant and an exponential function with positive base different from 1, or of

$(\dagger_3)$ the sum of a logarithm function [to a positive base different from 1] and a constant function, or of

$(\dagger_4)$ the product of a positive constant and a power function with nonzero exponent.

[For example, in the case of $(*_1)$, once it has been shown that f is monotonic and satisfies (1′), it follows from the first part of Theorem 221 that there is a number $c \neq 0$ such that [by $(*_1)$] for each $x \in \mathfrak{D}_g$ and $y \in \mathfrak{D}_g$,

$$g(x) - g(y) = c(x - y).$$

Choosing some argument x_0 of g, we see that

$$\forall_{x \in \mathfrak{D}_g}\ g(x) = c(x - x_0) + g(x_0).$$

Since $c \neq 0$, it follows that g is the linear function with slope c which contains $(x_0, g(x_0))$ restricted to $\mathfrak{D}_g$.]

To prove what has been asserted above, all that remains is to show, in each of the four cases, that if g is monotonic then so is f, and to show, for each case, under the appropriate hypothesis on $\mathfrak{D}_g$, that f satisfies the appropriate one of (1′), (2′), (3′), and (4′).

We begin by considering the first two cases, $(*_1)$ and $(*_2)$, in which we assume that $\mathfrak{D}_g$ is closed with respect to addition. As in all four cases, we assume that g is monotonic. Without loss of generality we may--and shall--assume that $\mathfrak{D}_f = \{u: \exists_{x \in \mathfrak{D}_g} \exists_{y \in \mathfrak{D}_g}\ u = x - y\}$. With this assumption it follows that if u_1 and u_2 are arguments of f then $u_1 = x_1 - y_1$ and $u_2 = x_2 - y_2$, where x_1, x_2, y_1, and y_2 are arguments

of g. And, since $u_1 + u_2 = (x_1 + x_2) - (y_1 + y_2)$, it follows that, since $\mathfrak{I}_g$ is closed with respect to addition, so is $\mathfrak{I}_f$. Moreover [with the same notation], $y_1 + y_2 \in \mathfrak{I}_g$, $u_1 + y_1 + y_2 = x_1 + y_2 \in \mathfrak{I}_g$, and $u_1 + u_2 + y_1 + y_2 = x_1 + x_2 \in \mathfrak{I}_g$. So if, in the case of $(*_1)$, '$\star$' denotes subtraction and, in the case of $(*_2)$, '$\star$' denotes division, we have, in either case,

$$f(u_1 + u_2) = g(u_1 + u_2 + x_1 + x_2) \star g(x_1 + x_2)$$

and $$f(u_2) = g(u_2 + x_1 + x_2) \star g(x_1 + x_2).$$

Hence, in either case,

$$\begin{aligned} f(u_1 + u_2) \star f(u_2) &= g(u_1 + u_2 + x_1 + x_2) \star g(u_2 + x_1 + x_2) \\ &= f(u_1) \end{aligned}$$

Consequently, in the case of $(*_1)$ ['$\star$' denoting subtraction],

$$f(u_1 + u_2) = f(u_1) + f(u_2)$$

and, in the case of $(*_2)$ ['$\star$' denoting division],

$$f(u_1 + u_2) = f(u_1)f(u_2).$$

So, in the first case, f satisfies (1′) and in the second it satisfies (2′). As to the monotonicity of f, since $u_2 - u_1 = (x_2 + y_1) - (x_1 + y_2) \in \mathfrak{I}_f$, it follows that, in either case,

$$f(u_2 - u_1) = g(x_2 + y_1) \star g(x_1 + y_2).$$

Suppose that $u_2 > u_1$--that is, suppose that $x_2 + y_1 > x_1 + y_2$. We are assuming that g is monotonic. Let's assume, more explicitly, that g is increasing. It follows that $g(x_2 + y_1) > g(x_1 + y_2)$ and, so, that in the case of $(*_1)$ that $f(u_2 - u_1) > 0$, while in the case of $(*_2)$ [in which the values of g are positive] that $f(u_2 - u_1) > 1$. In the first case, as we have proved,

$$f(u_2) = f(u_2 - u_1) + f(u_1)$$

So, since $f(u_2 - u_1) > 0$, it follows [for $u_2 > u_1$] that $f(u_2) > f(u_1)$. Hence, f is increasing and, in particular, is monotonic. In the second case, as we have proved,

$$f(u_2) = f(u_2 - u_1) \cdot f(u_1).$$

So, since $f(u_2 - u_1) > 1$ [and since, by $(*_2)$, $f(u_1) > 0$], it follows [for $u_2 > u_1$] that $f(u_2) > f(u_1)$. In this case, also, f is increasing and, hence, is monotonic. The alternative assumption--that g is decreasing--leads,

in the same way, to the conclusion that f is monotonic. Consequently, if g is monotonic and $\mathfrak{J}_g$ is closed with respect to addition then, in cases $(*_1)$ or $(*_2)$, f is monotonic and satisfies (1′) or (2′), respectively. This, as far as these cases are concerned, is what we set out to prove.

The cases $(*_3)$ and $(*_4)$ are handled by a minor reinterpretation of the preceding argument. In these cases the assumption about $\mathfrak{J}_g$ is that it is closed with respect to multiplication. If, everywhere in the preceding paragraph, we replace '-' by '/' and '+' by '•', the resulting argument shows, first, that in case $(*_3)$, f satisfies (3′) and, in case $(*_4)$, f satisfies (4′), and, second, that, in either case, if g is monotonic then so is f. This is what we needed to prove concerning cases $(*_3)$ and $(*_4)$.

BASIC PRINCIPLES AND THEOREMS

Commutative principles for addition and multiplication

$\forall_x \forall_y\ x + y = y + x$ $\qquad$ $\forall_x \forall_y\ xy = yx$

Associative principles for addition and multiplication

$\forall_x \forall_y \forall_z\ x + y + z = x + (y + z)$ $\qquad$ $\forall_x \forall_y \forall_z\ xyz = x(yz)$

Distributive principle [for multiplication over addition]

$\forall_x \forall_y \forall_z\ (x + y)z = xz + yz$

Principles for 0 and 1

$\forall_x\ x + 0 = x$ $\qquad$ $\forall_x\ x1 = x$ $\qquad$ $1 \neq 0$

Principle of Opposites

$\forall_x\ x + -x = 0$

Principle for Subtraction

$\forall_x \forall_y\ x - y = x + -y$

Principle of Quotients

$$\forall_x \forall_{y \neq 0}\ \frac{x}{y}y = x$$

* * *

1. $\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$ [page 2-60]
2. $\forall_x\ 1x = x$ [2-61]
3. $\forall_x \forall_a \forall_b \forall_c\ ax + bx + cx = (a + b + c)x$ [2-61]
4. $\forall_x \forall_y \forall_a \forall_b\ (ax)(by) = (ab)(xy)$ [2-61]
5. $\forall_x \forall_y \forall_a \forall_b\ (a + x) + (b + y) = (a + b) + (x + y)$ [2-61]
6. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow x + z = y + z]$ [2-64]
7. $\forall_x \forall_y \forall_z\ [x + z = y + z \Longrightarrow x = y]$ [2-65]
8. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow z + x = z + y]$ [2-66]

9. $\forall_x \forall_y \forall_z\ [z + x = z + y \Longrightarrow x = y]$ [page 2-66]

10. $\forall_x \forall_y\ [x = y \Longrightarrow -x = -y]$ [2-66]

11. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow xz = yz]$ [2-66]

12. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow zx = zy]$ [2-66]

13. $\forall_u \forall_v \forall_x \forall_y\ [(u = v \text{ and } x = y) \Longrightarrow u + x = v + y]$ [2-66]

14. $\forall_u \forall_v \forall_x \forall_y\ [(u = v \text{ and } u + x = v + y) \Longrightarrow x = y]$ [2-66]

15. $\forall_x\ x0 = 0$ [2-66]

16. $\forall_x \forall_y\ [x + y = 0 \Longrightarrow -x = y]$ [2-68]

17. $\forall_x\ --x = x$ [2-69]

18. $\forall_x \forall_y\ -(x + y) = -x + -y$ [2-69]

19. $\forall_x \forall_y\ -(x + -y) = y + -x$ [2-69]

20. $\forall_x \forall_y\ -(xy) = x \cdot -y$ [2-69]

21. $\forall_x \forall_y\ -(xy) = -xy$ [2-69]

22. $\forall_x \forall_y\ [x = -y \Longrightarrow -x = y]$ [2-69]

23. $\forall_x \forall_y\ -x \cdot -y = xy$ [2-70]

24. $\forall_x \forall_y\ -xy = x \cdot -y$ [2-70]

25. $\forall_x \forall_y \forall_z\ -x(y + z) = -(xy) + -(xz)$ [2-70]

26. $\forall_x \forall_y \forall_z\ -x(-y + -z) = xy + xz$ [2-70]

27. $\forall_x\ x \cdot -1 = -x$ [2-70]

28. $\forall_x\ -x = -1x$ [2-70]

29. $\forall_x \forall_y\ (x + y) + -y = x$ [2-71]

30. $\forall_x \forall_y\ (x + y) - y = x$ [2-71]

31. $\forall_x \forall_y \forall_z\ x - yz = x + -yz$ [page 2-71]

32. $\forall_x \forall_y\ x - y + y = x$ [2-72]

33. $\forall_x \forall_y\ -(x - y) = y - x$ [2-72]

34. $\forall_x \forall_y \forall_z\ x + (y - z) = x + y - z$ [2-73]

35. $\forall_x \forall_y \forall_z\ x - (y + z) = x - y - z$ [2-73]

36. $\forall_x \forall_y \forall_z\ x - (y - z) = x - y + z$ [2-73]

37. $\forall_x \forall_y \forall_z\ x + (y - z) = x - z + y$ [2-73]

38. $\forall_x \forall_y \forall_z\ x(y - z) = xy - xz$ [2-74]

39. $\forall_x \forall_y \forall_z\ (x - y)z = xz - yz$ [2-74]

40. $\forall_x \forall_y \forall_z\ x - (-y - z) = x + y + z$ [2-74]

41. $\forall_x \forall_y \forall_z \forall_u\ x - (y - z - u) = x - y + z + u$ [2-74]

42. $\forall_x\ 0 - x = -x$ [2-75]

43. $\forall_x\ x - 0 = x$ [2-75]

44. $\forall_x \forall_y \forall_z\ x + z - (y + z) = x - y$ [2-75]

45. $\forall_x \forall_y \forall_z\ x - z - (y - z) = x - y$ [2-75]

46. $\forall_a \forall_b \forall_c \forall_d\ (a - b) + (c - d) = (a + c) - (b + d)$ [2-75]

47. $\forall_x \forall_y \forall_z\ [z + y = x \Longrightarrow z = x - y]$ [2-89]

48. $\forall_x \forall_y \forall_{z \neq 0}\ [xz = yz \Longrightarrow x = y]$ [2-90]

49. $\forall_x \forall_{y \neq 0} \forall_z\ [zy = x \Longrightarrow z = \frac{x}{y}]$ [2-91]

50. $\forall_x\ \frac{x}{1} = x$ [2-91]

51. $\forall_{x \neq 0}\ \frac{x}{x} = 1$ [2-91]

52. $\forall_x\ \frac{x}{-1} = -x$ [2-91]

53. $\forall_{x \neq 0}\ \frac{0}{x} = 0$ [page 2-91]

54. $\forall_x \forall_{y \neq 0}\ [\frac{x}{y} = 0 \Longrightarrow x = 0]$ [2-91]

55. $\forall_x \forall_y\ [(x \neq 0 \text{ and } y \neq 0) \Longrightarrow xy \neq 0]$ [2-91]

56. $\forall_x \forall_y\ [xy = 0 \Longrightarrow (x = 0 \text{ or } y = 0)]$ [2-91]

57. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}$ [2-92]

58. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} - \frac{u}{v} = \frac{xv - uy}{yv}$ [2-92]

59. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv}$ [2-93]

60. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y}$ [2-94]

61. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} = \frac{x \div z}{y \div z}$ [2-95]

62. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y$ [2-96]

63. $\forall_x \forall_{y \neq 0}\ \frac{x}{y} = x \cdot \frac{1}{y}$ [2-97]

64. $\forall_x \forall_{y \neq 0}\ \frac{xy}{y} = x$ [2-97]

65. $\forall_{x \neq 0} \forall_y \forall_z\ \frac{xy + xz}{x} = y + z$ [2-97]

66. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \forall_{z \neq 0}\ \frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}$ [2-98]

67. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}$ [2-99]

68. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}$ [2-99]

69. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} - \frac{y}{z} = \frac{x - y}{z}$ [2-100]

70. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} - \frac{u}{vz} = \frac{xv - uy}{yvz}$ [2-100]

71. $\forall_x \forall_y \forall_{z \neq 0}\ x + \frac{y}{z} = \frac{xz + y}{z}$ [2-100]

72. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ x \div \frac{y}{z} = x\frac{z}{y}$ [page 2-101]

73. $\forall_x \forall_{y \neq 0} \forall_{u \neq 0} \forall_{v \neq 0}\ \frac{x}{y} \div \frac{u}{v} = \frac{xv}{yu}$ [2-101]

74. $\forall_{x \neq 0} \forall_{y \neq 0}\ \frac{1}{x/y} = \frac{y}{x}$ [2-101]

75. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} \div z = \frac{x}{yz}$ [2-101]

76. $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{-x}{y}$ [2-103]

77. $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{x}{-y}$ [2-103]

78. $\forall_x \forall_{y \neq 0}\ \frac{-x}{-y} = \frac{x}{y}$ [2-103]

79. $\forall_x\ [x \neq 0 \Longrightarrow -x \neq 0]$ [7-18]

80. $-0 = 0$ [7-18]

* * *

(P_1) $\forall_x\ [x \neq 0 \Longrightarrow$ either $x \in P$ or $-x \in P]$ [7-22]

(P_2) $\forall_x$ not both $x \in P$ and $-x \in P$ [7-22]

(P_3) $\forall_x \forall_y\ [(x \in P$ and $y \in P) \Longrightarrow x + y \in P]$ [7-23]

(P_4) $\forall_x \forall_y\ [(x \in P$ and $y \in P) \Longrightarrow xy \in P]$ [7-24]

* * *

81. $0 \notin P$ $[\forall_{x \in P}\ x \neq 0]$ [7-23]

82. $1 \in P$ $[1 > 0]$ [7-23]

* * *

(G) $\forall_x \forall_y\ [y > x \Longleftrightarrow y - x \in P]$ [7-30]

* * *

83. $\forall_x\ [x > 0 \Longleftrightarrow x \in P]$ [7-30]

84. $\forall_x \forall_y\ [y > x \Longleftrightarrow y - x > 0]$ [7-31]

85. $\forall_x\ [x < 0 \iff -x > 0]$ [page 7-32]

86 a. $\forall_x \forall_y\ [x \neq y \implies (x > y \text{ or } y > x)]$

b. $\forall_x \forall_y$ not both $x > y$ and $y > x$

c. $\forall_x \forall_y \forall_z\ [(x > y \text{ and } y > z) \implies x > z]$ [7-32]

d. $\forall_x \forall_y \forall_z\ [x > y \implies x + z > y + z]$

e. $\forall_x \forall_y \forall_z\ [(z > 0 \text{ and } x > y) \implies xz > yz]$

87. $\forall_x\ x \not> x$ $[\forall_x \forall_y\ (x = y \implies x \not> y)]$ [7-33]

88. $\forall_x \forall_y\ [y \geq x \iff x \not> y]$ [7-33]

89. $\forall_x \forall_y \forall_z\ [x + z > y + z \iff x > y]$ [7-33]

90. $\forall_x\ x + 1 > x$ [7-35]

91. $\forall_x \forall_y \forall_u \forall_v\ [(x > y \text{ and } u > v) \implies x + u > y + v]$ [7-35]

92. $\forall_x \forall_y \forall_z\ [(x > y \text{ and } y \geq z) \implies x > z]$ [7-35]

93. $\forall_x \forall_y\ [(x \geq y \text{ and } y \geq x) \implies x = y]$ [7-35]

94. $\forall_x \forall_y\ [-x > -y \iff y > x]$ [7-35]

95 a. $\forall_x \forall_y \forall_{z>0}\ [xz > yz \iff x > y]$

b. $\forall_x \forall_y \forall_{z<0}\ [xz < yz \iff x > y]$ [7-36]

96 a. $\forall_x \forall_y\ [xy > 0 \iff ([x > 0 \text{ and } y > 0] \text{ or } [x < 0 \text{ and } y < 0])]$

b. $\forall_x \forall_y\ [xy < 0 \iff ([x > 0 \text{ and } y < 0] \text{ or } [x < 0 \text{ and } y > 0])]$

[7-36]

97 a. $\forall_{x \neq 0}\ x^2 > 0$ [7-38]

b. $\forall_x \forall_y\ [x \neq y \implies x^2 + y^2 > 2xy]$ [7-39]

c. $\forall_{x > 0}\ x + \frac{1}{x} \geq 2$ [7-40]

98 a. $\forall_{x \geq 0} \forall_{y \geq 0} [x^2 = y^2 \Rightarrow x = y]$

b. $\forall_x \forall_{y \geq 0} [y^2 > x^2 \Rightarrow y > x > -y]$ [page 7-38]

c. $\forall_{x \geq 0} \forall_y [y > x \Rightarrow y^2 > x^2]$

99 a. $\forall_x \forall_y \forall_{z \neq 0} [\frac{x}{z} > \frac{y}{z} \Longleftrightarrow xz > yz]$ [7-41]

b. $\forall_{x \neq 0} ([\frac{1}{x} > 0 \Longleftrightarrow x > 0]$ and $[\frac{1}{x} < 0 \Longleftrightarrow x < 0])$

100. $\forall_{x > 0} \forall_{y \neq 0} \forall_{z > 0} [y > z \Rightarrow \frac{x}{z} > \frac{x}{y}]$ [7-41]

* * *

[domain of 'm', 'n', 'p', and 'q' is I^+]

(I_1^+) $1 \in I^+$

(I_2^+) $\forall_n\ n + 1 \in I^+$ [7-49]

(I_3^+) $\forall_S [(1 \in S$ and $\forall_n [n \in S \Rightarrow n + 1 \in S]) \Rightarrow \forall_n\ n \in S]$

* * *

101. $I^+ \subseteq P$ $[\forall_n\ n \in P]$ [7-49]

102. $\forall_m \forall_n\ m + n \in I^+$ [7-56]

103. $\forall_m \forall_n\ mn \in I^+$ [7-56]

104. $\forall_n\ n \geq 1$ [7-84]

105. $\forall_m \forall_n [n > m \Rightarrow n - m \in I^+]$ [7-84]

106. $\forall_m \forall_n [n \geq m + 1 \Longleftrightarrow n > m]$ [7-86]

107 a. $\forall_n\ n \not< 1$

b. $\forall_m \forall_n [n < m + 1 \Longleftrightarrow n \leq m]$ [7-86]

108. Each nonempty set of positive integers has a least member.
$[\forall_S [\emptyset \neq S \subseteq I^+ \Longrightarrow \exists_{m \in S} \forall_{n \in S}\ m \leq n]]$ [page 7-88]

* * *

(C) $\forall_x \exists_n\ n > x$ [7-89]

(I) $\forall_x [x \in I \Longleftrightarrow (x \in I^+ \text{ or } x = 0 \text{ or } -x \in I^+)]$ [7-94]

* * *

109. $\forall_x [x \in I \Longleftrightarrow \exists_m \exists_n\ x = m - n]$ [7-94]

*

[domain of 'i', 'j', and 'k' is I]

*

110 a. $\forall_j\ -j \in I$
b. $\forall_j \forall_k\ k + j \in I$
c. $\forall_j \forall_k\ k - j \in I$
d. $\forall_j \forall_k\ kj \in I$
[7-95]

111. $\forall_j \forall_k [k > j \Longleftrightarrow k - j \in I^+]$ [7-96]

112. $\forall_j \forall_k [k + 1 > j \Longleftrightarrow k \geq j]$ [7-96]

113. Each nonempty set of integers which has a lower bound has a least member. [7-98]

114. $\forall_j \forall_S [(j \in S \text{ and } \forall_{k \geq j} [k \in S \Longrightarrow k + 1 \in S]) \Longrightarrow \forall_{k \geq j}\ k \in S]$ [7-99]

115. Each nonempty set of integers which has an upper bound has a greatest member. [7-100]

116. $\forall_j \forall_S [(j \in S \text{ and } \forall_{k \leq j} [k \in S \Longrightarrow k - 1 \in S]) \Longrightarrow \forall_{k \leq j}\ k \in S]$ [7-100]

117. $\forall_S [(0 \in S \text{ and } \forall_k [k \in S \Rightarrow (k+1 \in S \text{ and } k-1 \in S)]) \Rightarrow \forall_k k \in S]$
[page 7-100]

* * *

$\forall_x \, [\![x]\!] = \text{the greatest integer } k \text{ such that } k \leq x$ [7-102]

$\forall_x \, \{\!\{ x \}\!\} = x - [\![x]\!]$ [7-107]

* * *

118 a. $\forall_x \forall_k \, [k \leq [\![x]\!] \Longleftrightarrow k \leq x]$ [7-103]

b. $\forall_x \forall_k \, [k > [\![x]\!] \Longleftrightarrow k > x]$ [7-104]

c. $\forall_x \forall_k \, [k \geq [\![x]\!] \Longleftrightarrow k + 1 > x]$

d. $\forall_x \forall_k \, [k < [\![x]\!] \Longleftrightarrow k + 1 \leq x]$

e. $\forall_x \forall_k \, [k = [\![x]\!] \Longleftrightarrow k \leq x < k + 1]$

(c–e) [7-105]

119. $\forall_x \forall_j \, [\![x + j]\!] = [\![x]\!] + j$ [7-105]

120. $\forall_x \forall_{y > 0} \exists_k \exists_z \, [x = ky + z \text{ and } 0 \leq z < y]$ [7-106]

121. $\forall_x \forall_{y > 0} \exists_n \, ny > x$ [7-106]

122. $\forall_x \, -[\![-x]\!] = \text{the least integer } k \text{ such that } k \geq x$ [7-106]

123. $\forall_x \forall_m \left(\left[\!\!\left[\frac{x}{m} \right]\!\!\right] = \left[\!\!\left[\frac{[\![x]\!]}{m} \right]\!\!\right] \text{ and } \left[\!\!\left[\left\{\!\!\left\{ \frac{x}{m} \right\}\!\!\right\} m \right]\!\!\right] = \left\{\!\!\left\{ \frac{[\![x]\!]}{m} \right\}\!\!\right\} m \right)$ [7-111]

124. $\forall_x \forall_m \; 0 \leq \left\{\!\!\left\{ \frac{[\![x]\!]}{m} \right\}\!\!\right\} m = [\![x]\!] - \left[\!\!\left[\frac{x}{m} \right]\!\!\right] m < m$ [7-111]

* * *

$\forall_m \forall_j \, [m \mid j \Longleftrightarrow \exists_k \, j = mk]$ [7-115 and 7-129]

* * *

125. $\forall_n \, (1 \mid n \text{ and } n \mid n)$ [7-115]

126 a. $\forall_m \forall_n [m|n \Rightarrow m \le n]$

b. $\forall_m \forall_n \forall_p [(m|n \text{ and } n|p) \Rightarrow m|p]$

c. $\forall_m \forall_n [(m|n \text{ and } n|m) \Rightarrow m = n]$

d. $\forall_m \forall_n \forall_p [(m|n \text{ and } m|p) \Rightarrow m|n + p]$ [page 7-115]

e. $\forall_m \forall_n \forall_p [(m|n \text{ and } m|n + p) \Rightarrow m|p]$

f. $\forall_m \forall_n \forall_p [m|n \Rightarrow mp|np]$ [7-116]

127. $\forall_m \forall_n \exists_i \exists_j \text{ HCF}(m, n) = mi + nj$ [7-122]

128. $\forall_m \forall_n \forall_k [(\text{HCF}(m, n) = 1 \text{ and } m|nk) \Rightarrow m|k]$ [7-129]

129. $\forall_m \forall_n [\text{HCF}(m, n) = 1 \Rightarrow$

$\forall_i \forall_j [mi + nj = 0 \Longleftrightarrow \exists_k (i = nk \text{ and } j = -mk)]]$ [7-129]

* * *

For each $j \in I$ and for each function a whose domain includes $\{k: k \ge j\}$,

$$\begin{cases} \displaystyle\sum_{i=j}^{j-1} a_i = 0 \\ \displaystyle\forall_{k \ge j-1} \sum_{i=j}^{k+1} a_i = \sum_{i=j}^{k} a_i + a_{k+1} \end{cases}$$

[page 8-36] [An earlier form is on page 8-9]

* * *

130. For any sequences a and b,

$$\left(b_1 = a_1 \text{ and } \forall_n \, b_{n+1} = b_n + a_{n+1}\right) \Rightarrow \forall_n \sum_{p=1}^{n} a_p = b_n$$ [8-17]

131 a. $\forall_n \sum_{p=1}^{n} 1 = n$ b. $\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

c. $\forall_n \sum_{p=1}^{n} p^2 = \frac{n(n+1)(2n+1)}{6}$ d. $\forall_n \sum_{p=1}^{n} p^3 = \frac{n^2(n+1)^2}{4}$ [page 8-24]

132 a. $\forall_n \sum_{p=1}^{n} 1 = n$ b. $\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

c. $\forall_n \sum_{p=1}^{n} p(p+1) = \frac{n(n+1)(n+2)}{3}$ [8-24]

d. $\forall_n \sum_{p=1}^{n} p(p+1)(p+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

133. $\forall_x \forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} xa_i = x \sum_{i=j}^{k} a_i$ [8-39]

134. $\forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} (a_i + b_i) = \sum_{i=j}^{k} a_i + \sum_{i=j}^{k} b_i$ [8-42]

135. $\forall_j \forall_{j_1 \geq j-1} \forall_{k \geq j_1} \sum_{i=j}^{k} a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^{k} a_i$ [8-44]

136. $\forall_j \forall_{k \geq j} \sum_{i=j}^{k} a_i = a_j + \sum_{i=j+1}^{k} a_i$ [8-44]

137. $\forall_j \forall_{j_1} \forall_{k \geq j-1} \sum_{i=j}^{k} a_i = \sum_{i=j+j_1}^{k+j_1} a_{i-j_1}$ [page 8-44]

138. $\forall_n \sum_{p=1}^{n} (a_{p+1} - a_p) = a_{n+1} - a_1$ [8-53]

139 a. $\forall_n \sum_{p=1}^{n} (p-1) = \frac{n(n-1)}{2}$ [8-55]

b. $\forall_n \sum_{p=1}^{n} (p-1)(p-2) = \frac{n(n-1)(n-2)}{3}$

c. $\forall_n \sum_{p=1}^{n} (p-1)(p-2)(p-3) = \frac{n(n-1)(n-2)(n-3)}{4}$

d. $\forall_n \sum_{p=1}^{n} (p-1)(p-2)(p-3)(p-4) = \frac{n(n-1)(n-2)(n-3)(n-4)}{5}$

[8-56]

* * *

$$\forall_p \ (\Delta a)_p = a_{p+1} - a_p$$ [8-57]

* * *

140. $\forall_n \ a_n = a_1 + \sum_{p=1}^{n-1} (\Delta a)_p$ [8-60]

* * *

A sequence a is an <u>arithmetic progression</u> if and only if the sequence Δa is a constant. The value of Δa is called the <u>common difference</u> of the AP. [8-66]

* * *

141 If a is an AP with common difference d, and, for each n, s_n is the sum of its first n terms, then

a. $\forall_n\ a_n = a_1 + (n-1)d$, b. $\forall_n \forall_{m \neq n}\ d = \dfrac{a_m - a_n}{m - n}$,

c. $\forall_n\ s_n = \frac{n}{2}[2a_1 + (n-1)d]$, d. $\forall_n\ s_n = \frac{n}{2}(a_1 + a_n)$. [page 8-67 and 8-68]

142. $\forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} a_i = \sum_{i=j}^{k} a_{k+j-i}$ [8-72]

143. $\forall_n \left[\forall_{m \leq n}\ a_m < b_m \Longrightarrow \sum_{p=1}^{n} a_p < \sum_{p=1}^{n} b_p \right]$ [8-76]

144. $\forall_x \forall_n \left[\sum_{p=1}^{n} a_p \geq x \Longrightarrow \exists_{m \leq n}\ a_m \geq \frac{x}{n} \right]$ [8-81]

* * *

For each $j \in I$ and for each function a whose domain includes $\{k: k \geq j\}$,

$$\begin{cases} \prod_{i=j}^{j-1} a_i = 1 \\ \forall_{k \geq j-1} \prod_{i=j}^{k+1} a_i = \prod_{i=j}^{k} a_i \cdot a_{k+1} \end{cases}$$ [8-94]

* * *

$$\forall_{k \geq 0}\ k! = \prod_{p=1}^{k} p$$ [8-98]

* * *

145. $\forall_j \forall_{k \geq j-1} \prod_{i=j}^{k} (a_i b_i) = \prod_{i=j}^{k} a_i \cdot \prod_{i=j}^{k} b_i$ [page 8-99]

146. $\forall_j \forall_{j_1 \geq j-1} \forall_{k \geq j_1} \prod_{i=j}^{k} a_i = \prod_{i=j}^{j_1} a_i \cdot \prod_{i=j_1+1}^{k} a_i$ [8-99]

147. $\forall_j \forall_{k \geq j} \prod_{i=j}^{k} a_i = a_j \cdot \prod_{i=j+1}^{k} a_i$ [8-99]

148. $\forall_j \forall_{j_1} \forall_{k \geq j-1} \prod_{i=j}^{k} a_i = \prod_{i=j+j_1}^{k+j_1} a_{i-j_1}$ [8-99]

149. $\forall_j \forall_{k \geq j-1} \prod_{i=j}^{k} a_i = \prod_{i=j}^{k} a_{k+j-i}$ [8-99]

* * *

$$\begin{cases} \forall_x \forall_{k \geq 0}\ x^k = \prod_{p=1}^{k} x & [8\text{-}100] \\ \forall_{x \neq 0} \forall_{k < 0}\ x^k = \dfrac{1}{x^{-k}} & [8\text{-}114] \end{cases}$$

* * *

150 a. $\forall_k\ 1^k = 1$ [8-103 and 8-115]

b. $\forall_k\ (-1)^{k+2} = (-1)^k$ [8-102]

c. $\forall_k\ [(-1)^{2k} = 1$ and $(-1)^{2k+1} = -1]$ [8-103]

d. $0^0 = 1$ and $\forall_n\ 0^n = 0$ [8-103]

151 a. $\forall_m \forall_{k \geq 0}\ m^k \in I^+$ [page 8-103]

b. $\forall_{m > 1} \forall_{k \geq 0}\ m^k > k$ [8-107]

152 a. $\forall_{x > 0} \forall_k\ x^k > 0$ b. $\forall_{x \neq 0} \forall_k\ x^k \neq 0$

c. $\forall_{x > 1} \forall_k\ x^{k+1} > x^k$

[8-103 and 8-115]

153. $\forall_x \forall_{k \geq 0}\ (x - 1) \sum_{p=1}^{k} x^{p-1} = x^k - 1$ [8-103]

154. $\forall_{x \neq 0} \forall_k\ x^{-k} = \dfrac{1}{x^k}$ [8-114]

155. $\forall_{x \neq 0} \forall_j \forall_k\ x^j x^k = x^{j+k}$ $[\forall_x \forall_{j \geq 0} \forall_{k \geq 0}\ x^j x^k = x^{j+k}]$ [8-117]

156. $\forall_{x \neq 0} \forall_j \forall_k\ \dfrac{x^j}{x^k} = x^{j-k}$ [8-118]

157. $\forall_{x \neq 0} \forall_j \forall_k\ (x^j)^k = x^{jk}$ $[\forall_x \forall_{j \geq 0} \forall_{k \geq 0}\ (x^j)^k = x^{jk}]$ [8-118]

158. $\forall_{x \neq 0} \forall_{y \neq 0} \forall_k\ (xy)^k = x^k y^k$ $[\forall_x \forall_y \forall_{k \geq 0}\ (xy)^k = x^k y^k]$ [8-119]

159. $\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\dfrac{x}{y}\right)^k = \dfrac{x^k}{y^k}$ $[\forall_x \forall_{y \neq 0} \forall_{k \geq 0} \left(\dfrac{x}{y}\right)^k = \dfrac{x^k}{y^k}]$ [8-119]

160. $\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\dfrac{x}{y}\right)^{-k} = \left(\dfrac{y}{x}\right)^k$ [8-119]

161. $\forall_{x > 0} \forall_j \forall_k \, [x^j = x^k \Longleftrightarrow (x = 1 \text{ or } j = k)]$ [page 8-122]

162. $\forall_{x \geq -1} \forall_{k \geq 0} \, (1 + x)^k \geq 1 + kx$ [8-125]

163. $\forall_{x > 1} \forall_y \forall_n \, [n \geq \frac{y}{x - 1} \Longrightarrow x^n > y]$ [8-125]

164. $\forall_{x \neq 0} \, [\frac{1}{x} > 1 \Longleftrightarrow 0 < x < 1]$ [8-125]

165. $\forall_{x \neq 1} \forall_{y > 0} \forall_n \left[\left(0 < x < 1 \text{ and } n \geq \frac{1}{y(1 - x)}\right) \Longrightarrow x^n < y\right]$ [8-126]

* * *

A sequence a is a <u>geometric progression</u> with <u>common ratio</u> r if and only if $a_1 \neq 0$ and $\forall_n \, a_{n+1} = a_n r$. [8-129 and 8-130]

* * *

166. $\forall_{x > 0} \forall_{y > 0} \, [x \neq y \Longrightarrow \frac{x + y}{2} > \sqrt{xy}]$ [8-131]

167. If a is a GP with common ratio r, and, for each n, s_n is the sum of its first n terms, then

a. $\forall_n \, a_n = a_1 r^{n-1}$,

b. for $r \neq 0$, $\forall_n \, \frac{a_{n+1}}{a_n} = r$,

c. for $r \neq 1$, $\forall_n \, s_n = \frac{a_1(1 - r^n)}{1 - r}$,

d. for $r \neq 1$, $\forall_n \, s_n = \frac{a_1 - a_n r}{1 - r}$.

[8-130 and 8-133]

168 a. For any GP, a, with common ratio r such that $|r| < 1$,

$$\sum_{p=1}^{\infty} a_p = \frac{a_1}{1-r}.$$

b. For each x such that $|x| < 1$, $\sum_{p=1}^{\infty} x^{p-1} = \frac{1}{1-x}$.

[pages 8-144 and 8-146]

* * *

$$\forall_{x \geq 0}\ |x| = x \text{ and } \forall_{x \leq 0}\ |x| = -x$$ [8-145]

* * *

169 a. $\forall_x \forall_y\ |x| \cdot |y| = |xy|$

b. $\forall_x \forall_y\ [|x| < y \Longleftrightarrow -y < x < y]$

c. $\forall_x \forall_y\ |x| - |y| \leq |x + y| \leq |x| + |y|$

[8-145]

170. $\forall_{k \geq 0} \forall_x \forall_y\ x^k - y^k = (x - y) \sum_{p=1}^{k} x^{k-p} y^{p-1}$ [8-160]

* * *

$$\begin{cases} \forall_{j \geq 0}\ C(j, 0) = 1 \\ \forall_{j \geq 0} \forall_{k \geq 0}\ C(j, k+1) = C(j, k) \cdot \frac{j-k}{k+1} \end{cases}$$ [8-168]

* * *

$$\begin{cases} 0! = 1 \\ \forall_{k \geq 0}\ (k+1)! = k! \cdot (k+1) \end{cases}$$ [8-169]

* * *

171. $\forall_{j \geq 0} \forall_{k \geq 0}\ C(j, k) = \dfrac{\prod_{i=0}^{k-1}(j - 1)}{k!}$ [page 8-169]

172. $\forall_{j \geq 0} \forall_{k \geq 0}\ C(j + k, k) = \dfrac{(j + k)!}{j!\,k!}$ [8-171]

173. $\forall_m \forall_n\ C(m, n) = C(m - 1, n) + C(m - 1, n - 1)$ [8-172]

* * *

(C_1) Two sets have the same number of members if and only if the members of one set can be matched in a one-to-one way with those of the other. [8-173]

(C_2) If no two of a family of sets have a common member then the number of members in the union of the sets is the sum of the numbers of members in the individual sets. [8-174]

(C_3) If a first event can occur in any of m ways, and, after it has occurred, a second event can occur in any of n ways, then the number of ways in which the two events can occur successively is mn. [8-177]

(C_4) If a set A is the union of n subsets, $A_1, A_2, \ldots, A_n$ and if N_1 is the sum of the numbers of members of the subsets, N_2 is the sum of the numbers of members of the intersections of the subsets two at a time, N_3 is the sum of the numbers of members of the intersections of the subsets three at a time, etc., then [8-193]

$$N(A) = \sum_{i=1}^{n} (-1)^{i-1} N_i .$$

* * *

174. $\forall_{j \geq 0} \forall_{k \geq 0}\ P(j, k) = \prod_{i=0}^{k-1} (j - i)$ [page 8-180]

175. The number of permutations of p things, of which p_1 are of a first kind, p_2 of a second kind, ..., p_n are of an nth kind, and the remainder are of different kinds, is [8-183]

$$\frac{p!}{\prod_{q=1}^{n} p_q!}.$$

176. $\forall_{j \geq 0}\ C_j = 2^j$ [8-185]

177. The number of odd-membered subsets of a nonempty set is the same as the number of its even-membered subsets. [8-185]

178. $\forall_x \forall_y \forall_{j \geq 0}\ (x + y)^j = \sum_{k=0}^{j} C(j, k) x^{j-k} y^k$ [8-197]

179 a. $\forall_m \forall_n \sum_{p=1}^{n} C(p - 1, m - 1) = C(n, m)$

b. $\forall_{k \geq 0} \forall_n \sum_{p=1}^{n} C(k + p - 1, k) = C(n + k, k + 1)$

[8-205]

180. For any sequence a whose mth difference-sequence is a constant, [page 8-206]

a. $\forall_n a_n = a_1 + \sum_{k=1}^{m} C(n-1, k)(\Delta^k a)_1$

and

b. $\forall_n \sum_{p=1}^{n} a_p = na_1 + \sum_{k=1}^{m} C(n, k+1)(\Delta^k a)_1$.

181. Each composite number n has a prime divisor p such that $p^2 \leq n$. [8-207]

182. For any sequence n of positive integers, and any prime number p, [8-214]

$$\forall_m [p \mid \prod_{i=1}^{m} n_i \Rightarrow \exists_{q \leq m}\, p \mid n_q].$$

183. Each positive integer other than 1 has a unique prime factorization. [8-213]

* * *

Least upper bound principle [ℓubp]:

> Each nonempty set [of real numbers] which has an upper bound has a least upper bound. [9-32]

[Note. With the adoption of this basic principle, basic principle (C) on page 9-344 becomes a theorem. See page 9-33.]

*

A function f is increasing on a set E if and only if $E \subseteq \mathfrak{D}_f$ and $\forall_{x_1 \in E} \forall_{x_2 \in E}[x_2 > x_1 \Rightarrow f(x_2) > f(x_1)]$.

An increasing function is one which is increasing on its domain.

A function f is decreasing on a set E if and only if $E \subseteq \mathfrak{D}_f$ and $\forall_{x_1 \in E} \forall_{x_2 \in E}[x_2 > x_1 \Rightarrow f(x_2) < f(x_1)]$.

A decreasing function is one which is decreasing on its domain.

A function is monotonic on a set if and only if it is either increasing on the set or decreasing on the set.

A function is monotonic if and only if it is either increasing or decreasing.

[9-35, 9-36, 9-37, and 9-194, 9-195]

* * *

184. Each monotonic function has a monotonic inverse of the same type. [9-37, 9-197]

185. $\forall_n \forall_{x_1 \geq 0} \forall_{x_2 \geq 0} [x_2 > x_1 \Rightarrow x_2^n > x_1^n]$ [9-37, 9-198]

185′. $\forall_n \forall_{x_1} \forall_{x_2} [x_2 > x_1 \Rightarrow x_2^{2n-1} > x_1^{2n-1}]$ [9-54]

* * *

A function f is continuous at x_0 if and only if $x_0 \in \mathfrak{D}_f$ and f(x) differs arbitrarily little from $f(x_0)$ for each $x \in \mathfrak{D}_f$ which is sufficiently close to x_0. [9-42]

[<u>Note</u>. For a more explicit definition, see page 9-211.]

A function is continuous if and only if it is continuous at each of its arguments. [9-42]

* * *

186. Each positive-integral power function is continuous [9-45, 9-218]

187. Each continuous monotonic function f whose domain is a segment $\overleftrightarrow{a, b}$ has a continuous monotonic inverse of the same type whose domain is the segment $\overleftrightarrow{f(a), f(b)}$. [9-45, 9-228]

* * *

(PR) $\forall_n \forall_{x \geq 0} \left(\sqrt[n]{x} \geq 0 \text{ and } (\sqrt[n]{x})^n = x\right)$ [9-49, 9-230]

* * *

188. $\forall_n \forall_{x \geq 0} \forall_y \left[(y \geq 0 \text{ and } y^n = x) \Rightarrow y = \sqrt[n]{x}\right]$ [9-49, 9-230]

189. Each principal positive-integral root function is continuous and increasing on the set of nonnegative numbers. [9-49, 9-230]

190. a. $\forall_m \sqrt[m]{0} = 0$ [9-51]

b. $\forall_m \forall_{x>0} \sqrt[m]{x} > 0$ [9-52]

c. $\forall_m \sqrt[m]{1} = 1$ [9-51]

191. a. $\forall_m \forall_{x \geq 0} \forall_{y \geq 0} \sqrt[m]{x}\sqrt[m]{y} = \sqrt[m]{xy}$

b. $\forall_m \forall_n \forall_{x \geq 0} \sqrt[m]{x} = \sqrt[nm]{x^n}$

c. $\forall_m \forall_n \forall_{x \geq 0} \sqrt[n]{\sqrt[m]{x}} = \sqrt[mn]{x}$

d. $\forall_j \forall_m \forall_{x>0} \left(\sqrt[m]{x}\right)^j = \sqrt[m]{x^j}$

[9-52]

* * *

(PR′) $\forall_n \forall_x \left(\sqrt[2n-1]{x}\right)^{2n-1} = x$ [9-55]

* * *

188′. $\forall_n \forall_x \forall_y \left[y^{2n-1} = x \Rightarrow y = \sqrt[2n-1]{x}\right]$ [9-55]

189′. Each principal odd positive-integral root function is continuous and increasing on the set of all real numbers. [9-55]

190′. $\forall_n \forall_x \sqrt[2n-1]{-x} = -\sqrt[2n-1]{x}$ [9-55]

191′. a. $\forall_{m \in O} \forall_x \forall_y \sqrt[m]{x}\,\sqrt[m]{y} = \sqrt[m]{xy}$ [O = set of odd positive integers]

b. $\forall_{m \in O} \forall_{n \in O} \sqrt[m]{x} = \sqrt[nm]{x^n}$

c. $\forall_{m \in O} \forall_{n \in O} \sqrt[n]{\sqrt[m]{x}} = \sqrt[mn]{x}$

d. $\forall_j \forall_{m \in O} \forall_{x \neq 0} (\sqrt[m]{x})^j = \sqrt[m]{x^j}$

[Note. For n even, $\forall_m \forall_x \sqrt[nm]{x^n} = \sqrt[m]{|x|}$,

For n even and m odd, $\forall_x \sqrt[nm]{x^n} = |\sqrt[m]{x}\,|$.] [9-56]

* * *

(R) $\forall_x [x \in R \iff \exists_n\, xn \in I]$ [9-62]

*

[domain of ‘r’, ‘s’, and ‘t’ is R]

* * *

192. a. $\forall_r\ -r \in R$ [9-65]

b. $\forall_r \forall_s\ r + s \in R$ [9-63]

c. $\forall_r \forall_s\ r - s \in R$

d. $\forall_r \forall_s\ rs \in R$

e. $\forall_r \forall_{s \neq 0}\ \frac{r}{s} \in R$ [9-65]

193. $\forall_n \forall_m \sqrt[n]{m}$ is irrational unless m is a perfect nth power. [9-65, 9-231]

* * *

A set is finite if and only if it is $\emptyset$ or, for some n, its members can be matched in a one-to-one way with those of $\{m: m \leq n\}$. [9-234]

A set is infinite if and only if its members can be matched in a one-to-one way with those of one of its proper subsets. [B is a proper subset of A if and only if $B \subseteq A$ and $B \neq A$.] [9-235]

A set is countably infinite if and only if its members can be matched in a one-to-one way with the positive integers. [9-237]

*

(C_5) $\left\{\right.$ A first set has a greater number of members than a second set if and only if members of the first set can be matched in a one-to-one way with all those of the second set and, no matter how this is done, there are members of the first set left over. [9-235]

* * *

194. All sets which are countably infinite have the same number of members, and a set which has the same number of members as some countably infinite set is, also, countably infinite. [9-237]

195. If each of two [disjoint] sets is countably infinite then so is their union. [9-237]

196. Each subset of a countably infinite set is either finite or countably infinite. [9-237]

197. Each Cartesian product of two countably infinite sets [as well as the Cartesian square of each countably infinite set] is countably infinite. [9-240]

198. The sets I^+, I, and R are countably infinite. [9-237]

* * *

A decimal of the form '$0.a_1a_2a_3\ldots$' represents the real number ℓ if and only if ℓ is the least upper bound of [9-244]

$$\{r: \exists_n\, r = \sum_{p=1}^{n} a_p \cdot 10^{-p}\}.$$

* * *

199. a. Each real number in $\overleftrightarrow{0,1}$ is represented by at least one decimal of the form '$0.a_1a_2a_3\ldots$', and each such decimal represents a unique real number in $\overleftrightarrow{0,1}$. [8-148, 9-244]

b. Two decimals of the form '$0.a_1a_2a_3\ldots$' represent the same real number if and only if, when the first place in which they differ is the nth, the nth digit numbers differ by 1 and the later digits in the decimal which has the larger nth digit-number are all '0's and those in the other decimal are all '9's. [9-245]

200. a. The set of all real numbers in $\overleftrightarrow{0,1}$ is not countably infinite.

b. There is a greater number of irrational numbers in $\overleftrightarrow{0,1}$ than there is of rational numbers altogether. [9-242]

201. $\forall_x \forall_{y>x} \exists_r\; x < r < y$ [9-67, 9-249]

* * *

$\forall_{x>0} \forall_r \forall_{m,\, rm \in I}\ x^r = \left(\sqrt[m]{x}\right)^{rm}$ [9-72]

$\forall_{x>1} \forall_u\ x^u =$ the least upper bound of $\{y: \exists_{r<u}\ y = x^r\}$

$\forall_{0<x<1} \forall_u\ x^u = (1/x)^{-u}$

$\forall_u\ 1^u = 1$ and $\forall_{u>0}\ 0^u = 0$

[9-92, 9-250]

* * *

202. For each $x > 0$, the exponential function with base x is a continuous function whose domain is the set of all real numbers; if $x \neq 1$, its range is the set of all positive numbers; it is decreasing if $0 < x < 1$ and increasing if $x > 1$. [9-93, 9-258]

203. $\forall_{x>0} \forall_u\ x^u > 0$ $\quad [\forall_{x\geq 0} \forall_{u\geq 0}\ x^u \geq 0]$ [9-75, 9-261]

204. $\forall_{x>0} \forall_u\ x^{-u} = 1/x^u$

205. $\forall_{x>0} \forall_u \forall_v\ x^u x^v = x^{u+v}$ $\quad [\forall_{x\geq 0} \forall_{u\geq 0} \forall_{v\geq 0}\ x^u x^v = x^{u+v}]$

206. $\forall_{x>0} \forall_u \forall_v\ x^u/x^v = x^{u-v}$

207. $\forall_{x>0} \forall_u \forall_v\ (x^u)^v = x^{uv}$ $\quad [\forall_{x\geq 0} \forall_{u\geq 0} \forall_{v\geq 0}\ (x^u)^v = x^{uv}]$

208. $\forall_{x>0} \forall_{y>0} \forall_u\ (xy)^u = x^u y^u$ $\quad [\forall_{x\geq 0} \forall_{y\geq 0} \forall_{u\geq 0}\ (xy)^u = x^u y^u]$

209. $\forall_{x>0} \forall_{y>0} \forall_u \left(\frac{x}{y}\right)^u = \frac{x^u}{y^u}$ $\quad [\forall_{x\geq 0} \forall_{y>0} \forall_{u\geq 0} \left(\frac{x}{y}\right)^u = \frac{x^u}{y^u}]$

210. $\forall_{x>0} \forall_{y>0} \left(\frac{x}{y}\right)^{-u} = \left(\frac{y}{x}\right)^{-u}$

211. $\forall_{x>0} \forall_u \forall_v\ [x^u = x^v \Rightarrow (x = 1 \text{ or } u = v)]$

* * *

(L) $\forall_{0 < a \neq 1} \forall_{x > 0}\ a^{\log_a x} = x$ [9-115]

* * *

212. $\forall_{0 < a \neq 1} \forall_{x > 0} \forall_y\ [a^y = x \Rightarrow y = \log_a x]$ [9-115]

213. The domain of each logarithm function is the set of positive numbers and its range is the set of real numbers. Each such function is continuous and monotonic--decreasing if its base is between 0 and 1 and increasing if its base is greater than 1. [9-117]

214. $\forall_{0 < a \neq 1}\ (\log_a 1 = 0 \text{ and } \log_a a = 1)$ [9-117]

215. $\forall_{0 < a \neq 1} \forall_{x > 0} \forall_{y > 0}\ \log_a(xy) = \log_a x + \log_a y$ [9-117]

216. $\forall_{0 < a \neq 1} \forall_{x > 0} \forall_{y > 0}\ \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ [9-118]

217. $\forall_{0 < a \neq 1} \forall_{x > 0} \forall_u\ \log_a(x^u) = u \cdot \log_a x$ [9-118]

218. a. $\forall_{0 < a \neq 1} \forall_{x > 0}\ \log_a(1/x) = -\log_a x$

b. $\forall_{0 < a \neq 1} \forall_{0 < b \neq 1}\ \log_b a = 1/\log_a b$

[9-118]

219. $\forall_{0 < a \neq 1} \forall_{0 < b \neq 1} \forall_{x > 0}\ \log_b x = \log_a x/\log_a b$ [9-131]

220. For each u, the power function with exponent u and positive arguments is continuous. [9-269]

[On pages 9-267 through 9-268 it is proved that products, reciprocals and composites of continuous functions are continuous.]

[Theorems 221-224 are used in section 9.10 and are proved in Appendix B, pages 9-313 through 9-336. We give, below, abbreviated statements of these four theorems.]

221. If f is a monotonic function such that

$$\forall_{x \in \mathcal{D}_f} \forall_{y \in \mathcal{D}_f} (x + y \in \mathcal{D}_f \text{ and } f(x + y) = f(x) + f(y)) \qquad [9\text{-}320]$$

then f is a subset of a homogeneous linear function.

222. If f is a monotonic function whose values are positive and is such that

$$\forall_{u \in \mathcal{D}_f} \forall_{v \in \mathcal{D}_f} (u + v \in \mathcal{D}_f \text{ and } f(u + v) = f(u)f(v)) \qquad [9\text{-}327]$$

then f is a subset of an exponential function with positive base different from 1.

223. If f is a monotonic function whose arguments are positive and is such that

$$\forall_{u \in \mathcal{D}_f} \forall_{v \in \mathcal{D}_f} (uv \in \mathcal{D}_f \text{ and } f(uv) = f(u) + f(v)) \qquad [9\text{-}330]$$

then f is a subset of a logarithm function.

224. If f is a monotonic function whose arguments and values are positive and is such that

$$\forall_{u \in \mathcal{D}_f} \forall_{v \in \mathcal{D}_f} (uv \in \mathcal{D}_f \text{ and } f(uv) = f(u)f(v)) \qquad [9\text{-}332]$$

then f is a subset of a power function with nonzero exponent.

TABLE OF SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1.000	3.162	51	2601	7.141	22.583
2	4	1.414	4.472	52	2704	7.211	22.804
3	9	1.732	5.477	53	2809	7.280	23.022
4	16	2.000	6.325	54	2916	7.348	23.238
5	25	2.236	7.071	55	3025	7.416	23.452
6	36	2.449	7.746	56	3136	7.483	23.664
7	49	2.646	8.367	57	3249	7.550	23.875
8	64	2.828	8.944	58	3364	7.616	24.083
9	81	3.000	9.487	59	3481	7.681	24.290
10	100	3.162	10.000	60	3600	7.746	24.495
11	121	3.317	10.488	61	3721	7.810	24.698
12	144	3.464	10.954	62	3844	7.874	24.900
13	169	3.606	11.402	63	3969	7.937	25.100
14	196	3.742	11.832	64	4096	8.000	25.298
15	225	3.873	12.247	65	4225	8.062	25.495
16	256	4.000	12.649	66	4356	8.124	25.690
17	289	4.123	13.038	67	4489	8.185	25.884
18	324	4.243	13.416	68	4624	8.246	26.077
19	361	4.359	13.784	69	4761	8.307	26.268
20	400	4.472	14.142	70	4900	8.367	26.458
21	441	4.583	14.491	71	5041	8.426	26.646
22	484	4.690	14.832	72	5184	8.485	26.833
23	529	4.796	15.166	73	5329	8.544	27.019
24	576	4.899	15.492	74	5476	8.602	27.203
25	625	5.000	15.811	75	5625	8.660	27.386
26	676	5.099	16.125	76	5776	8.718	27.568
27	729	5.196	16.432	77	5929	8.775	27.749
28	784	5.292	16.733	78	6084	8.832	27.928
29	841	5.385	17.029	79	6241	8.888	28.107
30	900	5.477	17.321	80	6400	8.944	28.284
31	961	5.568	17.607	81	6561	9.000	28.460
32	1024	5.657	17.889	82	6724	9.055	28.636
33	1089	5.745	18.166	83	6889	9.110	28.810
34	1156	5.831	18.439	84	7056	9.165	28.983
35	1225	5.916	18.708	85	7225	9.220	29.155
36	1296	6.000	18.974	86	7396	9.274	29.326
37	1369	6.083	19.235	87	7569	9.327	29.496
38	1444	6.164	19.494	88	7744	9.381	29.665
39	1521	6.245	19.748	89	7921	9.434	29.833
40	1600	6.325	20.000	90	8100	9.487	30.000
41	1681	6.403	20.248	91	8281	9.539	30.166
42	1764	6.481	20.494	92	8464	9.592	30.332
43	1849	6.557	20.736	93	8649	9.644	30.496
44	1936	6.633	20.976	94	8836	9.695	30.659
45	2025	6.708	21.213	95	9025	9.747	30.822
46	2116	6.782	21.448	96	9216	9.798	30.984
47	2209	6.856	21.679	97	9409	9.849	31.145
48	2304	6.928	21.909	98	9604	9.899	31.305
49	2401	7.000	22.136	99	9801	9.950	31.464
50	2500	7.071	22.361	100	10000	10.000	31.623

TABLE OF TRIGONOMETRIC RATIOS

Angle	sin	cos	tan	Angle	sin	cos	tan
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

Table of pairs (x, y) belonging to the inverse of the exponential function with base 10:

$\{(x, y): 10^y = x\}$

x	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010
x	0	1	2	3	4	5	6	7	8	9

x	0	1	2	3	4	5	6	7	8	9
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445
x	0	1	2	3	4	5	6	7	8	9

x	0	1	2	3	4	5	6	7	8	9
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996
x	0	1	2	3	4	5	6	7	8	9